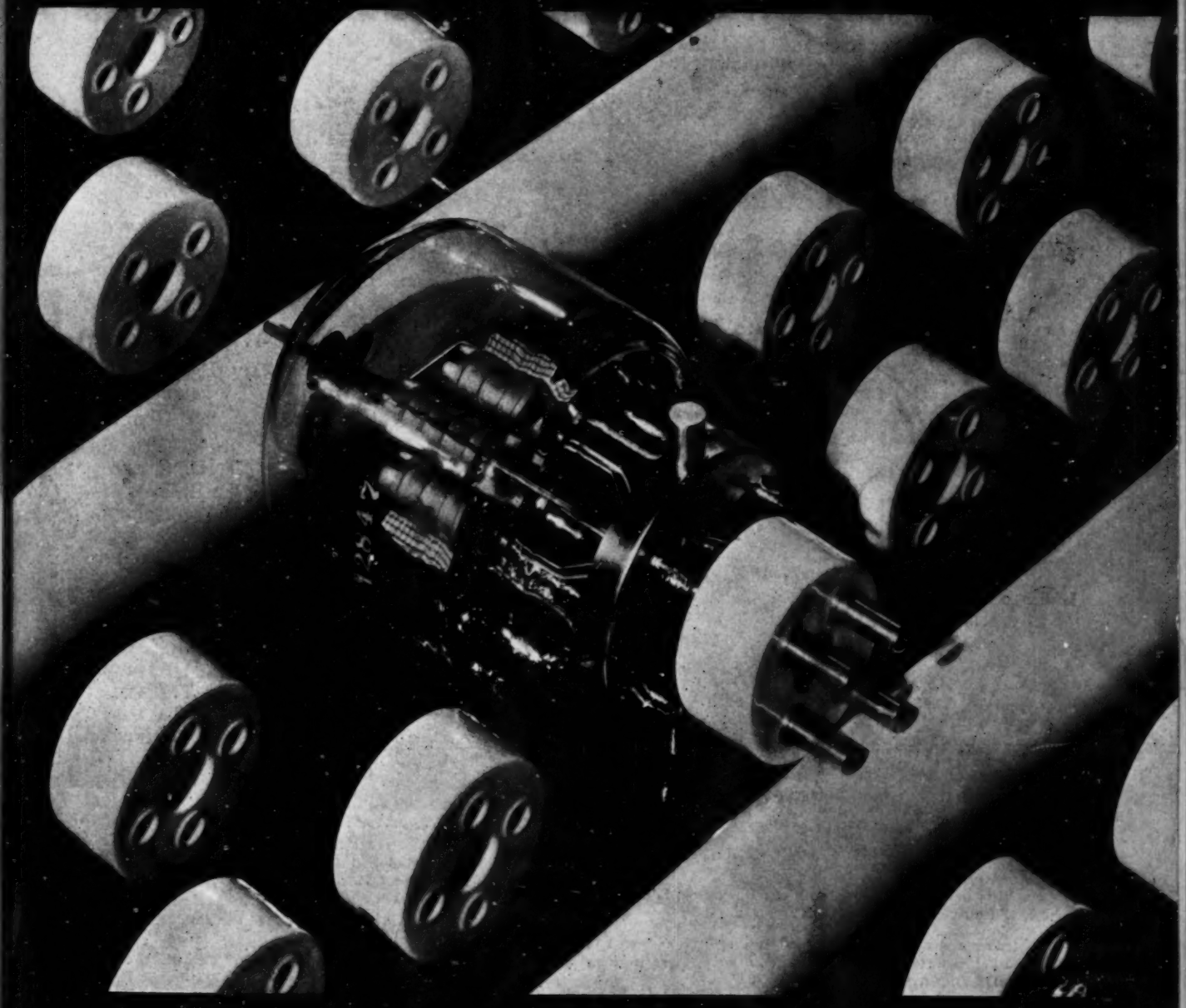
JOURNAL OF APPLIED PHYSICS



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New Books

Magnetic Circuits and Transformers

By The Electrical Engineering Department of Massachusetts Institute of Technology. Pp. 718, Figs. 384, 16×23½ cm. John Wiley & Sons, Inc., New York, 1943. Price $6.50.

This book, while primarily designed as a textbook for students, will, I am sure, be one which all transformer designers will wish to have readily available. The clarity of treatment and the well chosen examples and problems will make it of particular value to the young designer who does not have his magnetic fundamentals too well in hand. It fully meets the high quality which would be expected from the very able faculty at Massachusetts Institute of Technology. The choice of type size and printing are excellent. The diagrams and cuts are very clear and the illustrations well chosen.

The only typographical error noted is on page 66. The reference to Figure 12 should be to page 17 instead of page 15.

With reference to the choice of symbols, it is unfortunate that the practice of ASTM was not followed more closely, particularly since ASA will probably soon adopt practically the same symbols as ASTM. For instance: μ_0 should be used for initial permeability; μ_v for that of a vacuum; μ_Δ for incremental permeability; B_i is preferable to β for intrinsic induction; and $\mathfrak{F}$ is preferred for magnetomotive force. p_c is given in the table as "core loss per unit volume" but is used in the text as "core loss per unit mass."

In discussing the Epstein method of test, pages 146 and 147, it should be pointed out that corrections must be made for the losses in the voltmeter and potential circuit of the wattmeter, since these are always appreciable percentages of the measured loss.

In the discussion of eddy current losses, pages 132 to 140, it should be pointed out that the formulas, such as 28, do not give the correct eddy current losses even for commercial electrical sheet at 60 cycles. It is not uncommon for these losses to exceed the theoretical values by 100 percent or more for the higher silicon contents. Again, in the discussion of the separation of hysteresis and eddy current losses by the multiple frequency method, page 153, the intercept on the zero frequency axis will not in general check the hysteresis loss as measured ballistically, the latter value being usually low.

Referring to page 171, Figure 11-b, the dotted loop, as indicated, is doubtless a theoretical graph. An actual synthesis of a loop from the measured exciting current and induction will give a rounded tip due to eddy currents.

In Article 14, Chapter 6, I saw no mention of an important characteristic of displaced hysteresis loops, namely core losses. It is not uncommon for the core losses under incremental conditions and for a given B to be several times those under normal conditions.

Referring to Article 15-a, Chapter 6, it might have been well to call attention to the Hanna method for determining the air-gap, to give the maximum performance when dealing with incremental conditions. This method is widely used and very valuable. A reference is given in the bibliography—page 697.

In discussing current transformers, page 500, mention is made of the desirability of using high permeability low loss core materials. It might be pointed out that since the operating inductions are low, materials with high initial permeability are desired. When the nickel alloys were adopted in place of ordinary silicon steel for current transformer cores, the weight of core was greatly reduced and there was a corresponding reduction in the copper, resulting in lighter, smaller transformers with improved performance.

Thos. Spooner
Westinghouse Electric Company

Electromagnetic Waves

By S. A. Schelkunoff. Pp. 530+xv, Figs. 290, 16×23½ cm. D. Van Nostrand Company, Inc., New York, 1943. Price $7.50.

This book is certainly a very important contribution to the theory of electromagnetism, and will be heartily wel-

comed by the scientific public. No one better than Dr. Schelkunoff could undertake so bold an adventure as writing a new book on such a classical subject, and he undoubtedly presents us with a really new treatment of the whole theory, a very remarkable achievement in itself. This is the result of years of research in the field of electrical communications and radio, where the author patiently developed very precise theoretical discussions of problems of special importance for the engineers. As Dr. Schelkunoff states it, the book, as a whole, is an outgrowth of his consulting activities in Bell Telephone Laboratories, and due to his close association with experimentalists, the author was able to select a very large number of examples of great technical interest. It is a pleasure for the reader to find here, collected and connected together, the outstanding contributions of the author, and to follow clearly the very personal point of view which marks his whole research activities. The book itself took its final shape during the summer of 1942, when Dr. Schelkunoff was lecturing on electromagnetic waves at Brown University.

The first three chapters are devoted to a summary of the mathematics needed in the rest of the book: vector notations and coordinate systems, fundamental notions about oscillations and waves, and the most important properties of Bessel and Legendre functions. The mathematical methods are discussed in connection with practical problems of electrical engineering, and a number of important formulae have been collected in a very valuable section. Chapter 4 reviews the fundamental equations of electromagnetism, starting with a summary of the m. k. s. Giorgi unit system and going straight forward to the famous equations of Maxwell and their translation for waves of given frequency ω in different coordinate systems.

A physicist may be surprised, and even very much afraid, when he discovers that the author introduced systematically in his equations some additional terms of "magnetic charges and currents" in order to be able to write Maxwell's equations in a more symmetrical way. Such attempts have been made many times in the last fifty years, and always were met with suspicion and finally discarded. Of course, Dr. Schelkunoff is well aware of the danger, and explains very carefully that these additional terms have no physical meaning. This caution does not seem sufficient, as the whole history of modern science consistently proved the danger of such arbitrary assumptions, and how misleading they can be. The fact that such additional terms have been recently used by some authors, in connection with some problems of diffraction, is no sufficient reason, because it was not proved at all that this innovation is really necessary, and the only thing to be hoped is that a more correct treatment will soon be found, which will not make use of such fantastic assumptions.

In Chapter 5, entitled "Impedors, transducers, networks," the author presents the elements of circuit theory. There, the three-dimensional character of electromagnetic field is suppressed, and the discussion is conducted in terms of resistance, inductance, and capacitance. This provides the reader with the engineering background, and the essentials of the language into which the more elaborate theoretical results will later on be translated. One may, however, regret that the author did not take the opportunity to state more precisely the approximations and assumptions which lie at the basis of the simplified circuit theory, as could have been done, for instance, for the theoretical computation of a self-induction coefficient in terms of geometrical dimensions of the circuit. This discussion may be found scattered in different sections of the following chapters.

Chapter 6 deals with "Waves in general" or better said, sustained electromagnetic waves of given frequency, as the author mostly refers to this type of problem and only occasionally discusses waves of arbitrary time dependence. A number of special topics are treated under this general title: field of one or two current elements, of straight wires, loops, transmission lines and wave guides, single sheets or double layers, theory of images, and finally an extensive discussion of two-dimensional static problems.

Chapter 7 summarizes the classical theory of transmission lines. The main part of the book is contained in three dense chapters—8, 9, 10—called "Waves, wave guides, and resonators," "Radiation," and "Diffraction." It is difficult to give an idea of the very many practical problems discussed in these chapters; they include propagation in free space and waves guided inside hollow pipes of different shapes, as well as cavity resonators of all sorts. This is where the great experience of the author enables him to supply very valuable discussion of problems of great importance for the radio engineer. In Chapter 9, for instance, the reader will find a very complete discussion of radiation arrays containing a number of current elements or loops, and many applications of the theory of diffraction. It may be felt, however, that some important information is scattered in different examples, each one treated separately, rather than grouped for the convenience of the reader. This is, for instance, the case for the theory of the skin effect, which appears in a number of places.

Chapter 11 deals with the antenna theory, and is entirely based on the model of the "biconical antenna," a very interesting problem indeed, to which the author greatly contributed, discussing it with extreme skill, but there is only a very brief section on spherical antennas, practically nothing on the classical problem of spheroidal antennas, and the theory based on the use of retarded potentials (Oseen, Hallen, L. V. King, R. King, etc.) is completely ignored. The book concludes with a very brilliant discussion of the "impedance concept," one of the pet ideas of Dr. Schelkunoff and one which proved, especially in his hands, an extremely valuable method of discussion.

A very remarkable book, and full of information, it should be recommended to advanced readers, and to all theoreticians doing research on ultra-high frequency radio and its very numerous applications.

L. Brillouin
Brown University

Frontiers in Chemistry. Volume I. The Chemistry of Large Molecules

By H. Mark, E. O. Kraemer, A. Tobolsky, R. E. Powell, H. Eyring, R. M. Fuoss, C. S. Marvel, and E. Ott. Edited by R. E. Burk and O. Grummitt. Pp. 313+xiv, Figs. 60, 15×23 cm. Interscience Publishers, Inc., New York, 1943. Price $3.50.

This is the first of a series of volumes which represents an annual series of lectures at Western Reserve University, given under the title of Frontiers in Chemistry. The invited lecturers have been chosen as outstanding authorities in the field in which they lecture. The lecture is presented as a general review and summary of the topic assigned and each lecturer has been asked to submit a manuscript similar to the lecture material for publication in a collected volume. The series of lectures are arranged to cover a definite general topic, the first of which being on the Chemistry of Large Molecules.

(I) The Mechanism of Polyreactions, by H. Mark. A survey of the kinetics and steps involved in polyreactions.

(II) The Investigation of High Polymers with X-Rays, by H. Mark. The applications of x-ray data to the determination of the structure of synthetic and natural plastics and other high polymers.

(III) The Colloidal Behavior of Organic Macromolecular Materials, by E. O. Kraemer. The application of diffusion, dialysis, and viscosity properties to the determination of molecular character.

(IV) The Ultracentrifuge and Its Application to the Study of Organic Macromolecules, by E. O. Kraemer. The methods of sedimentation equilibrium, sedimentation velocity, and ultracentrifugal results.

(V) Elastic-Viscous Properties of Matter, by A. Tobolsky, R. E. Powell, and H. Eyring. The flow process as applied to long chain and other macromolecules.

(VI) The Electrical Properties of High Polymers, by R. M. Fuoss. The dielectric properties of plastics.

(VII) Organic Chemistry of Vinyl Polymers, by C. S. Marvel. The chemical reactions involved in the preparation of vinyl polymers.

(VIII) Chemistry of Cellulose and Cellulose Derivatives, by E. Ott. A survey of the chemical and physical properties of cellulose and some of its derivatives.

Wallace R. Brode
Ohio State University

Handbook of Elementary Physics

By Robert Bruce Lindsay. Pp. 382+xv, Figs. 96, 14½×21½ cm. The Dryden Press, New York, 1943. Price $2.25.

This is *not* a textbook in physics, but it is difficult to state in one sentence just what it *is*, for the word "Handbook" is sometimes subject to abuse. Unlike the well-known armloads of German books that frequently go by the name of "Handbuch," this small volume of 382 pages is portable but still too large to be palmed for examinations. It is designed to help the unfortunate student of physics "who can't get it all from the text" and who feels the need of supplementary assistance. But it is not a substitute for the text itself, despite its many admirable features; for no beginner could sip this distilled quintessence of physical principles without becoming tipsy unless he had first imbibed long draughts of the less intoxicating brew of the ordinary text. The thoughtful student can find great advantage in using this little volume as a reviewer-for-examinations and as an aid to a clearer understanding of what really matters in physics amid the many details of his study. It aims to train students in physical thinking, to help them understand how to do problems by comprehension rather than by mere substitution in formulas, to develop physical intuition by appeal to simple experiments not listed in the laboratory manual. In this respect, the author has adopted the attitude that there is a lot of good physics within the reach of every student by observation of everyday phenomena, and that a student who has his eyes open or who has his attention properly directed can see for himself the sort of thing that is *physics*.

The book is divided into three main parts. The first (130 pages) is called "A Primer of General Physics" and consists of an ultra-short text on the mechanical, thermal, electrical, and optical properties of matter; it contains no diagrams, no equations, and no definitions, but it is written in pithy, common-sense language and introduces some 66 simple "bathtub and automobile" experiments such as any young natural philosopher might like to perform. Here is the essential skeletal structure of physics by which its anatomy may be comprehended. In this section there are some 100 problems by which the student may test his own ability, and in many cases the author has made helpful hints on how to proceed, but always with emphasis on understanding rather than on substitution in formulas. In this section, the author has exhibited his own talents for exposition and elucidation, with a freshness and occasional bit of humor that makes good reading. For example, in dealing with the simplest aspects of motion, he offers the following: "Experiment I. Bring your car to a sudden stop and note that the occupants have a tendency to lunge forward. The remarks of those on the back seat about this experiment have little to do with physics and may be disregarded by the experimenter." Again, in describing a simple experiment with a garden hose, by which the experimenter may determine the very essence of projectile motion and the dependence of range upon initial angle, the author warns: "The last angle (90°) is not particularly recommended save for the most zealous researchers." Or, to take a third example, the experimenter may test Doppler's principle by having his friend drive past at high speed while blowing the horn of his car: "If he is like a good many people, he will enjoy this anyway, irrespective of zeal for science."

The second and third parts of the volume are a bit drier reading. The second part (135 pages) is a dictionary of terms, about 1000 in all, by which the first part may be translated into ordinary speech. No special comment is needed concerning these definitions, except to state that a liberal sampling of them shows them to be accurate and

adequate. The third part contains (1) a Chronological History of Physics (28 pages) arranged in tabular form with three parallel columns showing developments in physics, in other sciences, and contemporary political and social events culminating in the entry of the United States into World War II on Dec. 8, 1941; (2) a Bibliography of Suggested Reading with a list of 156 books under several categories: history, biography, philosophy, industrial applications, popular books in special fields, and physics in other branches of science; (3) a Collection of Useful Formulae (36 pages) involving only trigonometric and algebraic notation; (4) a Table of Physical Constants (15 pages); (5) some 20 pages of mathematical tables, including four-place trigonometric and logarithm tables; (6) answers to many of the problems in Part I.

Lindsay's *Handbook* is to the general physics text what a bouillion cube is to a roast of beef, and that includes the saltiness of the bouillion cube. For the many newcomers to the teaching of physics, and for those who want a "refresher," the book is particularly timely and suggestive. For the student who needs an extra lift not given by his own text, it *may* prove to be helpful provided such student will play the game with the author and try to learn physics as the author has here tried to teach it.

RICHARD M. SUTTON
Haverford College

The Mathematics of Physics and Chemistry

BY HENRY MARGENAU AND GEORGE M. MURPHY. Pp. 581, 16×24 cm. D. Van Nostrand Company, Inc., New York, 1943. Price $6.50.

The past few years have seen the appearance of several new textbooks in which are presented those branches of mathematics having direct application to the solution of problems in physics and engineering. These books have largely been directed toward the development of those methods and techniques which are useful in the solution of problems which might be described as of the classical type. Professors Margenau and Murphy have in the present book made a laudable attempt to fill a similar need of those interested in attacking problems in modern theoretical physics and chemistry or at least in acquiring sufficient mathematical facility to read modern literature in these fields. The principal emphasis in this book is on the development of those portions of mathematics which form the background of quantum mechanics, though, of course, most of the material has much wider application.

Chapter 1, entitled "The mathematics of thermodynamics," reviews the procedures of partial differentiation and their application to thermodynamics, and in addition contains a classification of thermodynamic relationships and a discussion of Caratheodory's principle.

"Ordinary differential equations," the second chapter, discusses the common methods of solution including a derivation of the solutions of Legendre's, Bessel's, Hermite's, Laguerre's, Mathieu's, and Gauss's hypergeometric equation and a discussion of Pfaff equations in view of their relationship to the material of the first chapter. Chapter 3, "Special functions," continues the analysis of the preceding chapter with a discussion of the analytical properties of the functions arising in the solution of the equations of Chapter 2 as well as others, the emphasis being placed on those properties which find repeated application in quantum mechanics and other fields. The chapter is marred by a totally inadequate treatment of complex variable theory of which free use is made throughout the chapter.

In the fourth chapter is contained a fairly complete development of the formalism of vector analysis together with a briefer venture into the analysis of tensors. This is applied in the following chapter to the consideration of the various coordinate systems most often occurring in physical problems. The result is a useful summary, not only for workers in quantum mechanics, but mechanics, acoustics, and electromagnetic theory as well.

A short account of the calculus of variations, comprising the sixth chapter, is adequate enough to indicate its application and general usefulness in both classical and quantum mechanics. In Chapter 7, entitled "Partial differential equations," the method of separation of variables is applied to obtain general and specific solutions of Laplace's equation, the equation of heat conduction, and the wave equation, and the use of Green's theorem in solving inhomogeneous equations is also indicated. The following chapter, "Eigenvalues and eigenfunctions," extends the discussion to eigenvalue problems as they arise in connection with boundary-value problems. The chapter, though brief, gives a useful and readable discussion of expansions in series of orthogonal functions and of the elements of Sturm-Liouville theory.

With a view toward applications to the theory of molecular spectra, the ninth chapter, "Mechanics of molecules," develops briefly the principles of classical mechanics and their use in the study of the rotation and vibration of molecules. The chapter on matrices which follows gives an excellent account of the algebra of matrices and develops with clarity the transformation properties of the various types of matrices which find recurrent application in quantum mechanics and group theory.

The book, in a sense, reaches a climax in a truly excellent one-hundred-page discussion of the principles of nonrelativistic quantum mechanics. Hardly any phase of the subject is inadequately treated—the axiomatic foundations, the Schrodinger equation, the matrix mechanics, approximate methods of solving eigenvalue problems, time-dependent perturbations, electron spin, the many-body problem, and the exclusion principle are all not only touched upon, but discussed often in considerable detail and with multitudinous examples. The whole chapter is written with such logic and clarity that it is well worth reading even by advanced students in quantum mechanics.

In the following chapter are developed the general principles of statistical mechanics, both from the classical point of view of Gibbs and the quantum viewpoint of Darwin and Fowler. Chapter 13 discusses various methods of numerical calculation which are useful in connection

with the application of many of the mathematical techniques discussed in the book.

The two final chapters in the book are excellent discussions of integral equations and group theory, respectively, again with particular emphasis on their applications to quantum mechanical problems. The chapter on group theory is particularly readable and contains adequate discussions of particular groups and the theory of group characters.

Whenever such a large amount of material as above is compressed between the covers of a single volume, something must be sacrificed, and in mathematics it is usually rigor which meets this fate. Considering the spatial limitations, the authors have maintained a reasonable standard in general, though on occasion the lack of rigor is sufficient either to make questionable the validity of some of the mathematical processes, or to make obscure the conditions under which they are applicable.

Perhaps most striking about the book is the enthusiastic spirit of the authors in wanting very much to "put over" the fundamental ideas and techniques of modern physics so that no one need be denied an insight into the truly amazing developments in theory of the present century. The book, with its collection of problems and examples, can well be used as a text for a senior or first-year graduate course in the mathematical techniques of modern physics, or as a source of this material for self-study. It will, no doubt, find a worthy place not only in the libraries of students but of many active workers in physics and theoretical chemistry.

LESLIE L. FOLDY
Columbia University, Division of War Research

Infinite Series

BY J. M. HYSLOP, Pp. 120+xi, 12½×19 cm. Interscience Publishers, Inc., New York, 1942. Price $1.75.

This book is intended for second- or third-year students who have some knowledge of the principles of elementary analysis. The title *Infinite Series* is not warranted by the contents of the book which gives a fairly complete treatment of the more limited field of the theory of *convergence* of *real* series. The treatment, which is arithmetical, will appeal primarily to the student interested in pure mathematics.

Some of the more interesting topics in the field, such as transformation of series, asymptotic series, operations with divergent series, etc., are not taken up at all, presumably because of the limited size of this series of mathematical texts. It is precisely for this deficiency that this reviewer hesitates to recommend the book as a text for students in applied mathematics. To this latter type of student the standard rules of convergence of series are of significance only to the extent to which he is taught how and when to dodge them.

No references are given to other textbooks on infinite series. This is unfortunate, since an introduction of the beginner to a classic such as Bromwich's theory of *Infinite Series*, to name but one, is as important as acquainting him with any of the theorems on convergence.

C. L. PEKERIS
Columbia University, Division of War Research

Applied Electronics

By members of the Staff of the Department of Electrical Engineering, Massachusetts Institute of Technology. Pp. 772+xxiii, Figs. 342, 15×23 cm. John Wiley & Sons, Inc., New York, 1943. Price $6.50.

Applied Electronics is the third volume of the important *Principles of Electrical Engineering Series* which has dealt thus far with *Electric Circuits* and *Magnetic Circuits and Transformers*. This series is designed primarily to provide texts for advanced engineering students, but the close attention which has been given to fundamentals will make these volumes, particularly the present one, a valuable addition to the library of physicists engaged in electrical research.

This volume is described in its subtitle as "A first course in electronics, electron tubes, and associated circuits." It falls naturally into three parts. The first part, comprising one hundred and fifty-two pages, deals with the pertinent physical phenomena of electron ballistics in electric and magnetic fields, electron emission, and conduction through vacuum, gases, and vapors. Ninety-three pages are then devoted to the design and characteristics of vacuum tubes and gas tubes. The remainder of the book treats the applications of vacuum and gas tubes in electric circuits, and includes discussions of single and polyphase rectifiers, class *A*, *AB*, and *C* amplifiers, oscillators, modulation, and detection.

Chapter I acquaints the reader with the motion of charged particles in electrostatic and magnetostatic fields. The basic equations of motion are set up and electron paths are deduced for several cases, including some simple electron lenses, the secondary emission multiplier, and a hypothetical space charge free magnetron.

Chapter II opens with a succinct outline of the behavior of electrons in solids, quoting the form of the Fermi-Dirac energy distribution and deducing therefrom the thermionic emission equation. The troublesome *A* factor is dealt with firmly and correctly. Tungsten, thoriated tungsten, and oxide-coated thermionic emitters each receive several pages of detailed description including, for example, Jones and Langmuir's table of the characteristics of ideal tungsten filaments. Sections on Schottky effect, field emission, photoelectric emission, and secondary emission make up the rest of the chapter. The section secondary emission, as well as sections elsewhere in the book on similarly controversial subjects, is, perhaps, open to mild criticism on the basis that it fails to impress the reader with the diversity of experimental result and opinion to be found in the literature. Without emphasis on this disagreement, it is dangerous to hand the student a graph of secondary emission against primary energy.

Chapter III, on conduction through vacuum and gases, includes a deduction of the 3/2 power law and a discussion of the limitations it imposes on conduction through space.

The effect of small amounts of gas is then considered, and the student is thus led painlessly into the complex field of ionization and collision phenomena and gas discharges.

Chapters IV and V, on vacuum tubes and gas tubes, respectively, apply the principles already laid down to the constants and characteristics of diodes, triodes, tetrodes, pentodes, gas rectifiers, ignitrons, and thyratrons. Examples of each are given, but no tabulation of practical tubes is attempted. *The Radio Amateurs Handbook* is, of course, available for anyone desiring such a tabulation; nevertheless, such advances in the electronic art as the 6AC7 and the VR series seem, to this reviewer, to merit inclusion by name in a text of this sort.

One hundred and thirty-one pages are devoted to the theory and operation of single and polyphase rectifiers in Chapters VI and VII. Standard rectifier, filter, and control circuits are described. These two chapters seem to have been written by a power engineer. The regulated power supply, standard among radio engineers, is not mentioned.

The last five chapters, approximately the last half of the book, make up a pellucid and complete treatment of the use of vacuum tubes as circuit elements. The methods of equivalent circuits are set up and applied to a great number of important cases. The usual amplifier considerations are supplemented by analyses of such devices as push pull amplifiers, inverse feedback amplifiers including the cathode follower, and class B and C amplifiers. The last chapter, on modulation and detection, studies the modulation process in detail with particular attention to the methods of modulating class C amplifiers. Linear, square law and heterodyne detection are described, and the book concludes with a discussion of frequency modulation that is a model of clarity. Phase modulation is mentioned, but the subtle considerations which render its use impractical are passed over rather lightly. Television techniques, as such, go untreated in this text—there is, for instance, no discussion of the iconoscope.

Such sins of omission as have been mentioned can scarcely be laid at the authors' feet, however. In the words of their preface "This book is for a first basic course. Rigor of thought and analysis rather than extensiveness of scope is its intended feature." This objective has been achieved to a high degree. Moreover, as also is noted in the preface, the authors' output was impounded by the war effort before they were able to deal with all of the subjects which had originally been scheduled for inclusion.

The text is illustrated by more than one hundred and fifty problems of sufficient practical interest that they cannot safely be neglected by the conscientious student. The reader is given a thorough introduction to m.k.s. units which are used throughout. The notation conforms to that adopted in 1938 by the Institute of Radio Engineers. In some cases this results in a slightly cumbersome and unfamiliar symbolism. For example, the I.R.E. has seized upon the symbol "e" to represent instantaneous voltage. The charge on the electron must then be represented otherwise—in this case by "Q_e."

It is pleasing to detect, in every section of this book, the hand of a major authority in his respective field. The text throughout reflects the high pedagogical and technical skill which is to be expected from the staff of the Massachusetts Institute of Technology.

JOHN P. BLEWETT
General Electric Company

Here and There

The Electron Microscope Society of America

The Chicago Section of the American Chemical Society arranged to sponsor a symposium on electron microscopy at the National Chemical Exposition held in Chicago during the week of November 23, 1942. Dr. G. L. Clark, University of Illinois, was appointed Chairman of a Committee composed of Dr. Clark, Dr. L. A. Matheson of Dow Chemical Company, and Dr. O. S. Duffendack of the University of Michigan, to make the arrangements for this conference which was scheduled to be held in the Sherman Hotel on November 27 and 28.

In attendance at this two-day conference were some seventy or eighty scientists of the country who are actually doing work in the field of electron microscopy. This conference held three eminently successful meetings and as a result a need was immediately recognized for a formation of a society which would assure that similar conferences be held in the future. Under the guidance of this temporary committee, a resolution was adopted by those in attendance that such a society be formed and the group of officers were elected: President, Dr. R. Bowling Barnes; Vice President, Dr. Albert F. Prebus; Secretary-Treasurer, M. C. Banca; Directors, V. K. Zworykin and O. S. Duffendack. Also, the society was to be tentatively named the National Conference on Electron Microscopy.

These officers have met and discussed in detail the future of this newly formed society. It was decided that the society be called the Electron Microscope Society of America and by unanimous ballot of all members the title has become fixed. This society was formed to advance the science of electron microscopy. Its aims are to provide means for distributing technical information through proposed publications, to distribute reprints of technical work to those interested, to attempt a complete bibliography, and to provide opportunities for workers in this field to meet and discuss their problems, techniques, and results. Its further aims are to set up representative groups which will attempt to develop standards which will permit all scientists and technicians in the various fields in which electron microscopy might be employed to make use of a uniform and accurate terminology and to establish suitable test procedures so that the results of development workers will be directly comparable.

In order that suitable meetings might be arranged, at which, in addition to the presentation of papers, symposia on certain specific phases of this work might be held, the society recognizes the many benefits which it would derive from affiliation with some other already established national scientific organization. In view of the fact that

electron microscopes are already being applied successfully to practically every branch of science and industry, it was difficult to know to which national organization to turn. After several discussions, it was decided that the greatest advantage would result from affiliation with two well-known organizations: namely, the American Institute of Physics and the American Association for the Advancement of Science. We are pleased to report that at this time the Electron Microscope Society of America is now an Associated Society of the American Institute of Physics and contacts have been made for the affiliation with the American Association for the Advancement of Science. By thus affiliating with these two groups, the society will still continue to function as an independent organization and at the same time will receive all of the advantages of being a part of these national groups.

The society has made tentative plans for its first meeting sometime in January of 1944, to be held in conjunction with a possible meeting of the American Physical Society in New York.

Quarterly of Applied Mathematics

A new *Quarterly of Applied Mathematics* has been founded primarily to meet the needs of mathematicians and engineers whose interests extend beyond the accepted boundaries of their respective groups. This periodical is published under the sponsorship of Brown University and prints original papers in applied mathematics which have an intimate connection with industry or practical science.

The tables of contents of the first two issues of this journal are given below:

April, 1943

Foreword
Tooling Up Mathematics for Engineering. TH. V. KÁRMÁN.
A Review of the Statistical Theory of Turbulence. H. L. DRYDEN
On the Motion of a Pendulum in a Turbulent Fluid. C. C. LIN.
On Plane Rigid Frames Loaded Perpendicularly to Their Plane. W. PRAGER AND G. E. HAY.
On the Vibrations of a Clamped Plate Under Tension. A. WEINSTEIN AND W. Z. CHIEN.
A Direct Image Error Theory. M. HERZBERGER.
The Impedance of a Transverse Wire in a Rectangular Wave Guide. S. A. SCHELKUNOFF.
Notes:
On A. C. Aitken's Method of Interpolation. W. FELLER.
A New Derivation of Munk's Formulae. W. C. RANDELS.
Concerning the Acceleration Potential. L. BERS.

July, 1943

Forced Vibrations of Systems with Non-Linear Restoring Force. K. O. FRIEDRICHS AND J. J. STOKER.
On the Deflection of Anisotropic Thin Plates. V. MORKOVIN.
Symmetrical Joukowsky Airfoils in Shear Flow. H. S. TSIEN.
On the Discontinuous Flow Around an Airfoil with Flap. P. W. KETCHUM.
On a Class of Differential Equations in Mechanics of Continua. L. BERS AND A. GELBART.
Book Reviews.

Fall Courses at Polytechnic Institute

The Electrical Engineering Department of the Polytechnic Institute of Brooklyn announces a new series of graduate courses under the ESMWT program: Introduction to microwave theory, Introduction experiments in microwaves, Theory of cathode-ray circuits, Experiments in cathode-ray circuits, and Ultra-high frequency electronics.

Interchemical Review

The *Interchemical Review* is a quarterly publication of the Research Laboratories of Interchemical Corporation, 432 West 45 Street, New York, New York.

The tables of contents of the first two issues are given below:

Spring, 1943

Molecular Architecture of Resins. ALBERT G. CHENICEK.
Glossary of Rheology. A Staff Report.
Patents You May Not Know. THEODORE POST.
Interchemical News Notes:
Technical Articles and Addresses by I. C. Men.
In the Service.
Contributors to This Issue.
IPI Essay Contest Details Announced.
Aridye Makes Flameproof Colors for Camouflage.

Summer, 1943

Infrared Photography. FREDERICK B. GLEASON, JR.
Women in Science. HELEN O. VECHOR.
The Diminutive Diatom. E. F. FULLAM AND MARY SCHUSTER.
Gremlins, Chemlins, and Demons.
Interchemical News Notes:
Technical Papers by I. C. Authors.
In the Service.
Contributors to This Issue.
N. Y. Boy Wins IPI Contest.

Third Nation-Wide Science Talent Search

The third annual Science Talent Search, a nation-wide quest for promising scientific ability among high school graduating seniors, will be conducted during the fall and winter months of the school year just opened, Watson Davis, Director of the Science Clubs of America, announced in Washington. Open alike to boys and girls, the Science Talent Search will enlist the aid of high school principals and teachers who will administer tests and supply other necessary data. Westinghouse Science Scholarships totaling $11,000 will be awarded to at least 10, and possibly 40, of the young scientists of the future. In addition, all 40 will attend a five-day Science Talent Institute on an all-expense trip to Washington next February.

Final examinations during sessions of the Institute will determine the award of two four-year Westinghouse Science Grand Scholarships of $2,400 each and eight four-year Westinghouse Science Scholarships of $400 each. One boy and one girl will be selected to receive the Grand Scholarships. The Westinghouse Electric & Manufacturing Company provides the scholarship awards as a contribution to the advancement of science in America. Additional Westinghouse scholarships totaling $3,000 will be awarded at the discretion of the judges.

If any scholarship recipient should be drafted or enter other government war service, his scholarship will be held for his use later.

Staff Changes

Expansion of activities has necessitated the addition of five new members to the staff of Foster D. Snell, Inc. Harold Moore Schiller, B.S., Yale University, formerly with the Norda Essential Oil and Chemical Company, and Arthur Streuli, B.S., recent Lehigh University graduate have been assigned to the Physiological and Organic Department. Henry A. Tisch, M.S., New York University, formerly with the American Chicle Company has joined the Physical Chemistry Department. Robert W. Pease, B. S., Worcester Polytechnic Institute, formerly with Calco Chemical Company, is now doing research in the Rubber and Plastics Department. The Special Research Department has added to its staff Ida Borowich, B.A., recent Hunter College graduate.

Need for Equipment in the War Effort

Urgent requests for instruments difficult or impossible to obtain through customary channels continue to reach the Committee on Location of New and Rare Instruments. Many of these can be filled and vital research projects greatly helped if you will do your bit. Particularly needed are sensitive electrical instruments (milli- and micro-volt and ammeters) and usable optical instruments or parts. Please go over your instruments, and if you have any that you can spare, send a list of them to: D. H. Killeffer, Chairman, 60 East 42 Street, New York 17, New York. Instruments are requested for both loan and outright purchase.

Requests in the hands of the Committee which remain unfilled are:

Pressure autoclaves
Cenco impulse counters
Metallurgical microscopes
Polarizer and analyzer for microscopes
Abbe refractometers (16 requests)
Strobotacs
L. & N. portable thermocouples.

Instruments available through the Committee include:

Various balances
S. & H. colorimeters
L. & H. H-ion meter
Microtomes
*p*H apparatus
Polarimeters
Potentiometers
Pyrometers
Hilger-Spekker spectrophotometer viscosimeters.

Bulletin of Mathematical Biophysics

The December, 1943 issue of *The Bulletin of Mathematical Biophysics*, edited by N. Rashevsky and published by the University of Chicago Press, will contain the following articles:

A Logical Calculus of the Ideas Immanent in Nervous Activity. WARREN S. MCCULLOCH AND WALTER PITTS.

A Statistical Consequence of the Logical Calculus of Nervous Nets. H. D. LANDAHL, W. S. MCCULLOCH, AND WALTER PITTS.

A Mathematical Theory of the Affective Psychoses. JEROME Y. LETTVIN AND WALTER PITTS.

On Synchronous Sporulation with Possible Reference to Malarial Parasites. ALSTON S. HOUSEHOLDER.

A Theory of the Induced Size Effect. ALSTON S. HOUSEHOLDER.

Note on Rashevsky's Equation for Cellular Growth. ALSTON S. HOUSEHOLDER.

On the Origin of Life. N. RASHEVSKY.

Index to Volume Five.

Institute Devoted to Research in Glass

Dr. W. A. Weyl, Professor of Glass Technology at Pennsylvania State College announced that an institute devoted to fundamental research in glass is now in process of formulation. At his invitation, representatives of leading companies interested in the development and use of glass, met at the Hotel Biltmore, New York on August 17 and selected an organizing committee to prepare a charter for the new institute.

The original sponsors of this institute are: American Optical Company, Bausch and Lomb Optical Company, Corning Glass Works, Eastman Kodak Company, and Pittsburgh Plate Glass Company. It is expected that membership in the new institute will be open to all concerns interested in glass research. The institute will establish laboratories whose location will be decided upon at a later meeting of the organizing committee.

New Officers of the National Association of Science Writers

Newly elected officers of the National Association of Science Writers are: President, Robert D. Potter, science editor of *The American Weekly;* and Secretary-Treasurer, Captain Stephen J. McDonough, AUS, Office of the Surgeon General of the Army. Committee appointments include Howard Blakeslee, the Associated Press, chairman of the President's Advisory Committee; William Laurence, *The New York Times*, chairman of the War Advisory Committee; Gobind B. Lal, International News Service, chairman of the Program Committee. A special publication committee has been appointed to explore the possibilities of founding an official journal of the association—a magazine designed to bring the advances of science and medicine to a wide layman audience. Serving on the committee with Potter, Laurence, and Blakeslee are Waldemar Kaempffert, *The New York Times;* David Dietz, Scripps-Howard Newspapers, and Lawrence Salter, the American Medical Association.

Cover Photograph

A new ceramic plastic called "Prestite" is used as bases for high frequency radio tubes and is manufactured by Heintz & Kaufman, Ltd., South San Francisco, California. Prestite combines the electrical and mechanical strength of wet process porcelain with the moulding qualities of dry process porcelain and provides a substitute for critical materials. It is formed under heavy hydraulic pressure that imparts a dense grain structure, enabling it to stand more electrical, mechanical, and chemical abuse than the average ceramic. It is used in many products where intricate shapes must meet high insulation requirements. Photograph has been supplied by the Westinghouse Electric and Manufacturing Company.

Contributed Original Research

Rheological Properties of Colloidal Solutions, Pigment Suspensions, and Oil Mixtures

R. N. Weltmann and H. Green
Interchemical Corporation, New York, New York

(Received August 18, 1943)

The law of Arrhenius which correlates the viscosity of a colloidal solution logarithmically to the volume percent concentration of solid matter has previously been tested only for dilute suspensions which appeared to the investigators to behave like true Newtonians. Similar suspensions have been tested at higher concentrations of solid matter and the authors found that although Arrhenius' law could not be checked, a similar exponential law evolved, which, however, is correct only within the tested region of viscosity. The authors realized that with the increase of solid matter the suspension can become plastic or pseudoplastic and may also show thixotropic behavior. Therefore it was considered as a major part of this investigation to determine the validity of Arrhenius' law in that region of concentration, where plasticity occurs. Although Arrhenius' law was not found to be applicable for plastic materials, two exponential laws, similar to Arrhenius' law, have been established between the plastic viscosity and the volume percent of pigment content of a plastic suspension on the one hand, and between the yield value and the volume percent of pigment content of a plastic suspension on the other hand. The two constants in the exponents of the two equations have been shown to be logarithmically related to the average diameter, d_3, of the pigment particles contained in the suspension. Oil mixtures with the same and with different type constituents have been tested below a certain rate of shear, called "the limiting rate of shear," where they are Newtonian liquids, and also above this limiting rate of shear where they behave like thixotropic plastics. Their Newtonian viscosities, obtained at rates of shear below the limiting rate of shear, have been found to increase logarithmically with the volume percent of one of the oils contained in the mixture, which is in agreement with Arrhenius' exponential law. However, above the limiting rate of shear, oil mixtures deviate from the exponential law, which is logically expected, since their limiting rate of shear depends on the Newtonian viscosity and therefore is different for each oil mixture.

THE investigation of viscosities of colloidal solutions and suspensions is one of the classical problems of rheology. As early as 1887 Arrhenius[1] worked on this problem and found that the viscosities of colloidal solutions and suspensions increased logarithmically with the concentration of solid matter in the liquid. Arrhenius formulated his law as follows:

$$\eta = \eta_0 e^{kc} \tag{1}$$

where η is the viscosity of the suspension, η_0 is the viscosity of the solvent, c is the concentration in terms of grams of solute per cc of solvent, and k is a constant characterized by the solute, solvent, and temperature.

This fundamental law has been, in course of time, discussed and essentially accepted by many investigators. A great number of solutions and suspensions have been tested, but the literature does not indicate that the different researchers considered the fact that such suspensions may greatly change their flow characteristics, if the solid matter is increased above a certain percentage.

If a suspension consists of two different constituents of which one may be a true Newtonian liquid like water, while the other one is a solid of particular particle size and shape, then it must be expected that as a result of the increased percentage of solid matter and as a function of the geometry of the particles the flow characteristics will be affected. The suspension which, with low solid matter content may yet effectively be a Newtonian liquid, will with the percentage increase of the solid constituent change to a plastic material or to a pseudoplastic. In other words,

[1] Arrhenius, Zeits. f. physik. Chemie 1, 285 (1887).

the suspension will exhibit a yield value, if a plastic; or a varying viscosity with rate of shear, if a pseudoplastic. The dilute suspension, if it is a Newtonian liquid, develops flow at the smallest shearing force and its viscosity is independent of rate of shear; whereas the plastic suspension produced by the higher percent content of solid matter requires a minimum shearing force, i.e., the yield value, to overcome the internal friction, before flow can start. As soon as flow starts the plastic viscosity is again independent of rate of shear. Should the suspension become pseudoplastic it will flow at the smallest shearing force but its viscosity decreases continuously with increasing rate of shear.

The existence of plasticity or pseudoplasticity of a suspension must not be overlooked by the investigator. Viscosity measurements can only be accurately conducted if one takes the presence of yield value or pseudoplasticity into consideration. It means that a viscosity determination obtained from only one rate of shear is not satisfactory, but the proper viscosity values must be determined for each concentration by a series of measurements using different rates of shear. Figure 1 shows the characteristic relationship between rate of shear and shearing stress of a Newtonian liquid, of a plastic, and of a pseudoplastic material. The Newtonian viscosity in one case and the plastic viscosity in the other is proportional to cot α. The yield value is proportional to the intercept on the shearing stress axis. In the case of a pseudoplastic, the viscosity decreases continuously with increasing rate of shear. The initial viscosity is defined as being proportional to cot α.

In some cases such a suspension with high solid content will not only act like a plastic or pseudoplastic, but will in addition exhibit thixotropy. Thixotropic materials will undergo an isothermal gel-sol-gel transformation upon agitation and subsequent rest. Therefore, an experimental procedure must be followed, which makes a logical comparison possible of the viscosities at different concentrations, if the suspensions are thixotropic.

The authors wished to determine whether the fundamental logarithmic law of Arrhenius is also correct in that region where the suspension acts as a plastic material. Apparently this never has been investigated before.

Previous workers have been satisfied in checking Arrhenius' law either by using dilute concentrations or by making sure the suspensions behaved like true Newtonians at the rates of shear tested.

Bredée and Booys[2] have summarized the various investigations in this field and their paper is to be highly recommended for any one who desires to obtain a comprehensive survey and bibliography of that work. Reference should also be made to other papers. Traxler, Schweyer, and Mofatt[3] have shown that a linear relation exists between the logarithm of the viscosity of viscous mixtures made of an asphalt and a pulverulent solid and the volume percent of the solid present. Verman[4] proved that the exponential law holds

[2] Bredée and Booys, Kolloid Zeits. **79**, 31/49 (1937).
[3] Traxler, Schweyer, and Mofatt, Ind. Eng. Chem. Ind. Ed. **29**, 489 (1937).
[4] Verman, London Shellac Research Bureau, Technical Paper No. 11 (July, 1937).

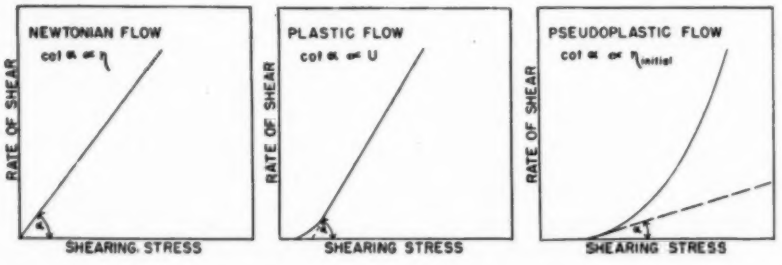

FIG. 1. The relationship between rate of shear and shearing stress shown schematically for a Newtonian liquid and a plastic and pseudoplastic material.

true for lac solutions. Berl and Buetler[5] and later Duclaux and Wollman[6] have shown that nitrocellulose solutions follow a similar exponential law.

The authors of this paper have investigated resin, ethyl cellulose, nitrocellulose, and chlorinated rubber solutions at higher solid content than previously tested. They have taken the precaution to determine the viscosities at low rates of shear but have not relied on a one-point measurement of viscosity. All of these solutions acted effectively as true Newtonians at low rates of shear, at which the viscosity measurements have been obtained. The results found are shown in Fig. 2 and Table I. By plotting the logarithm of viscosity against solid concentration straight lines have been obtained in the tested region of viscosity. Thus an exponential law evolved which, although similar to Arrhenius' law does not check it. According to Arrhenius, see Eq. (1), the intercept on the logarithmic viscosity axis should be equal to the viscosity of the solvent. However, this was not found to be the case for the straight lines, Fig. 2, of the colloidal solutions described in Table I. Most of these straight lines intercept the logarithmic viscosity axis at a point below the viscosity of the solvent contained in the suspension, indicating that either the lines eventually curve towards the viscosity axis at lower solid content, or, if Arrhenius' law is correct, that the lines will sharply change their direction at a certain low percent of solid content. Hence, these curves, Fig. 2, might be approximations and straight only within the tested region of viscosity.

The more interesting work relates to plastic materials, where similar conditions have been encountered. Although various investigators[7–10] have made measurements of plastic viscosities and yield values of suspensions at different pigment concentrations, they did not correlate their results in such a manner that would either prove or disprove the correctness of Arrhenius' law when applied to plastic materials. The authors of this paper conducted a great number of experiments to see if it was possible to decide this point. Plastic viscosities and yield values at a given shearing rate together with the corresponding volume concentration of solid matter were measured.

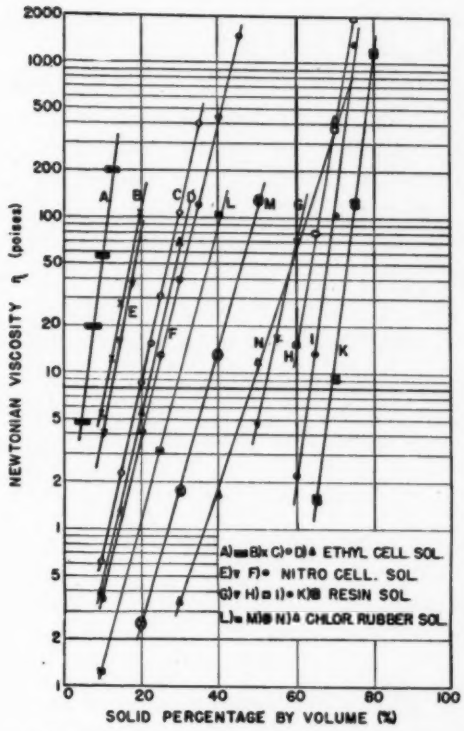
NEWTONIAN VISCOSITY η (poises)
SOLID PERCENTAGE BY VOLUME (%)
A) B) C) D) ETHYL CELL SOL.
E) F) NITRO CELL. SOL.
G) H) I) K) RESIN SOL.
L) M) N) CHLOR RUBBER SOL.

FIG. 2. Newtonian viscosities of various solutions as a function of solid concentration by volume.

TABLE I. Composition of the materials shown in Fig. 2.

No.	Solid	Solvent
A	3 poises ethyl cellulose	80% toluene and 20% butanol
B	0.44 poise ethyl cellulose	80% toluene and 20% butanol
C	0.07 poise ethyl cellulose	80% toluene and 20% butanol
D	0.07 poise ethyl cellulose	ethyl acetate
E	5/6" nitrocellulose	ethyl acetate
F	1/2" nitrocellulose	ethyl acetate
G	Paranol resin	20% ketone and 80% naphtha
H	Amberol resin	toluene
I	Zitro resin	toluene
K	Cumar resin	toluene
L	1.53 poises chlorinated rubber	toluene
M	0.23 poise chlorinated rubber	toluene
N	0.05 poise chlorinated rubber	toluene

EXPERIMENTAL PROCEDURE

To determine the plastic viscosity and yield value, a rotational viscometer was used, which

[5] Berl and Buetler, Zeits. f. Gesamte Schiess und Sprengstoffwesen **5**, 82 (1910).
[6] Duclaux and Wollman, Bull. de la Soc. Chim. **27**, 414 (1920).
[7] Green and Haslam, Ind. Eng. Chem. **19**, 1, 53 (1927).
[8] Mardles, J. Oil Colour Chem. Assn. **25**, 267 (1942).
[9] Sheets, Paper Trade J. **16**, 3 (1943).
[10] Wolarowitch and Tolstoi, Kolloid Zeits. **70**, 165 (1935).

has been described in detail in a paper by Green[11] and more briefly in other papers.[12–15]

In conducting the experiments the rate of shear applied to the material under investigation was gradually increased to a predetermined value and then gradually decreased. This decrease must follow immediately the increase in rate of shear, since otherwise thixotropic breakdown will occur because of the application of the top rate of shear over a period of time. This has been described in another paper.[15] Thus flow curves are obtained, which have been introduced in earlier papers.[11–14] A typical flow curve of a thixotropic plastic is shown in Fig. 3. Two down curves $A\tau_{2(A)}$ and

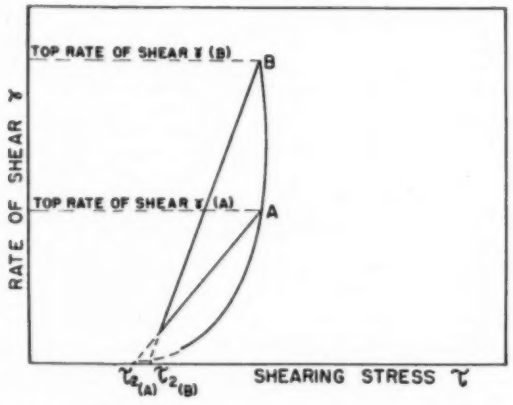

FIG. 3. Schematic flow curve of a thixotropic plastic material for two different top rates of shear.

$B\tau_{2(B)}$ are shown representing the thixotropic levels of the material for their respective top rates of shear. From the linear portion of the down curves, the plastic viscosity and yield value can be obtained from Reiner's[16] equation as follows:

$$U=S'(\tau-\tau_2)/\gamma \qquad (2)$$

$$f=C'\tau_2 \qquad (3)$$

where U is the plastic viscosity in poises, f is the yield value in dynes per square centimeter, τ is the shearing stress in dynes per square centimeter, τ_2 is the shearing stress at the intercept the straight part of the down curve produces on the torque axis; this is also measured in dynes per square centimeter, γ is the rate of shear measured in sec.$^{-1}$; and S' and C' are instrumental constants.

EXPERIMENTAL RESULTS

Plastic Viscosity as Function of Pigment Concentration

In Figs. 4 and 5 the logarithm of the plastic viscosity is plotted against the volume percent of solid content for various pigment suspensions. Each straight line in Fig. 4 represents a different pigment suspended in a linseed oil of a true Newtonian viscosity of about 3.9 poises. All these suspensions have been measured at a top rate of shear of 150 sec.$^{-1}$. The equation which can be obtained from the straight lines in Figs. 4 and 5 is:

$$U=(\eta_0+A)e^{Bp} \qquad (4)$$

where U is the plastic viscosity of the pigment suspension, η_0 is the Newtonian viscosity of the oil; p is the volume percentage of the pigment, and A and B are constants of the material. These constants, shown in Table II, probably depend on particle shape, particle size, force of flocculation, and on the top rate of shear to which the material has been measured.

TABLE II. The material constants A, B, M, and N of Eqs. (4) and (5) for the various pigment suspensions tested.

	Constant top rate of shear =150 sec.$^{-1}$ Constant temp. =30°C Oil viscosity η_0 =3.9 poises			
Pigment Suspended in a Linseed Oil	A	B	M	N
Carbon Black	−0.6	0.186	12.5	0.434
Leaded Zinc Oxide No. 1	+2.1	0.091	132	0.060
Green Seal Zinc Oxide	+2.1	0.080	330	0.072
Basic Carb. White Lead No. 1	+0.9	0.077	0.86	0.188
Antimony Oxide	−1.2	0.087	7.6	0.114
Basic Carb. White Lead No. 2	−0.7	0.075	60	0.080
Fine Zinc Oxide	+4.5	0.080	96	0.150
White Seal Zinc Oxide	+6.6	0.066	180	0.083
Leaded Zinc Oxide No. 2	+3.4	0.071	150	0.062
Titanium Dioxide No. 1	−1.2	0.103	20	0.101
Titanium Dioxide No. 2	−3.15	0.147	17.5	0.111
Chrome Yellow	−0.5	0.082	400	0.074

To investigate A and B in relation to the particle size of the pigment, three pigments (a zinc oxide and two high leaded zinc oxides) were reheated and grown to different particle sizes, as has been done in previous work.[7] Thus particle

[11] Green, Ind. Eng. Chem. Anal. Ed. **4**, 576 (1942).
[12] Weltmann, Ind. Eng. Chem. Anal. Ed. **15**, 424 (1943).
[13] Green and Weltmann, Ind. Eng. Chem. Anal. Ed. **3**, 201 (1943).
[14] Fischer and Jerome, Ind. Eng. Chem. Ind. Ed. **35**, 336 (1943).
[15] R. Weltmann, J. App. Phys. **14**, 343 (1943).
[16] Reiner and Riwlin, Kolloid Zeits. **43**, 1 (1927).

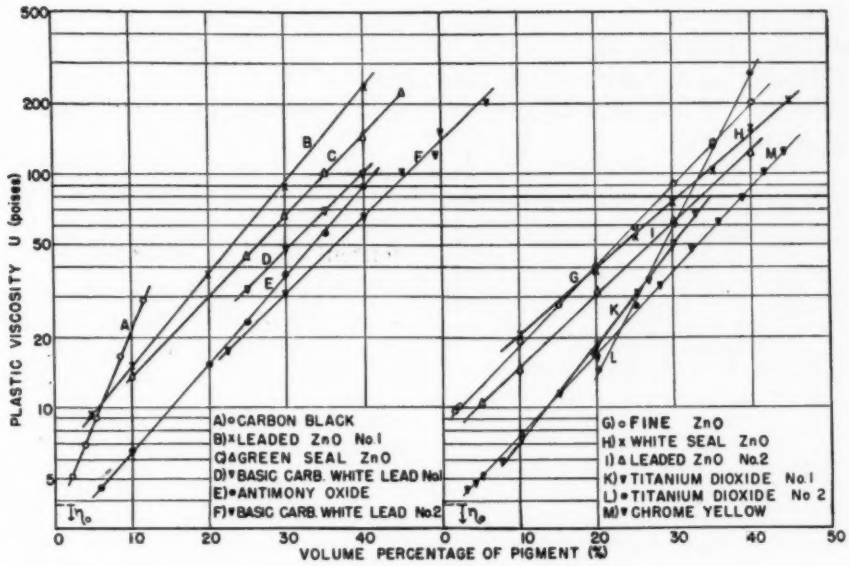
PLASTIC VISCOSITY U (poises)
VOLUME PERCENTAGE OF PIGMENT (%)
A) CARBON BLACK
B) LEADED ZnO No.1
C) GREEN SEAL ZnO
D) BASIC CARB. WHITE LEAD No.1
E) ANTIMONY OXIDE
F) BASIC CARB. WHITE LEAD No.2
G) FINE ZnO
H) WHITE SEAL ZnO
I) LEADED ZnO No.2
K) TITANIUM DIOXIDE No.1
L) TITANIUM DIOXIDE No.2
M) CHROME YELLOW

FIG. 4. The logarithm of plastic viscosity as a function of solid concentration by volume for pigment suspensions.

shape and force of flocculation are believed to be closely the same for all heated pigments within a series. All straight lines in Fig. 5 are obtained from one of the leaded zinc oxides. The average particle diameter with respect to specific surface, designated in the literature[17] as d_3, was measured by the microscopical method for the three reheated series of pigments. In correlating the average particle diameter, d_3, with the constants A and B, it was found that a logarithmic relationship exists between d_3 and B, Fig. 6 (Table III), indicating that B (the slope) definitely depends

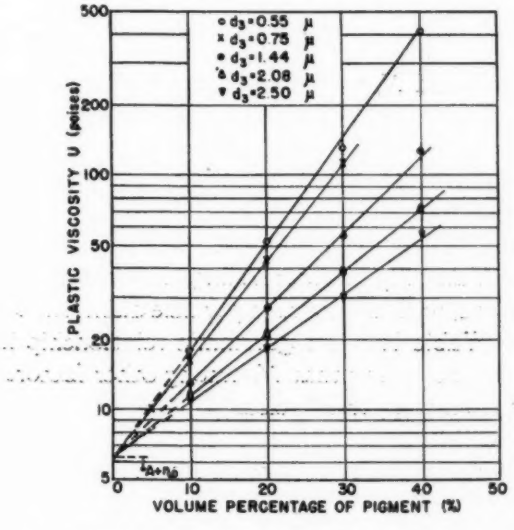

FIG. 5. The logarithm of plastic viscosity as a function of solid concentration by volume for various suspensions of a leaded zinc oxide of varying average particle diameter, d_3.

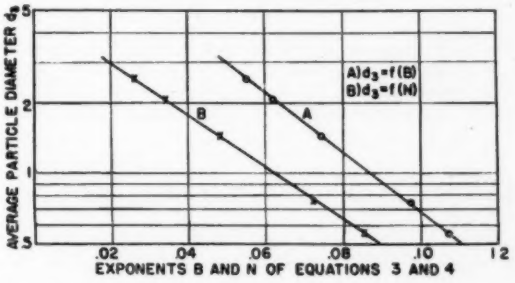

FIG. 6. The logarithm of the average particle diameter d_3 as a function of the exponent B of Eq. (4), curve A, and as a function of the exponent N of Eq. (5), curve B.

TABLE III. The average particle diameter, d_3, of a leaded zinc oxide heated to various temperatures, in relation to the material constants A, B, M, and N of Eqs. (4) and (5).

Constant top rate of shear = 150 sec.$^{-1}$
Constant temp. = 30°C
Oil viscosity η_0 = 3.9 poises

Particle diameter, d_3 (microns)	A	B	M	N
0.55	2.3	0.107	140	0.085
0.75	2.3	0.097	137	0.072
1.44	2.3	0.074	129	0.048
2.08	2.3	0.062	126	0.034
2.50	2.3	0.055	95	0.026

[17] Green, J. Frank. Inst. **204**, 713 (1927).

on the particle size of the pigment. As seen from Fig. 5 and Table III, A (the intercept in excess of η_0) seems to be independent of the particle size, since it is practically the same for all the zinc oxides regardless of their average particle diameters.

Comparing Eq. (4) with Arrhenius' law (Eq. (1)) the difference between these two exponential equations becomes obvious. Whereas, according to Arrhenius' law, the intercept on the logarithmic viscosity axis is equal to the viscosity of the liquid phase, in Eq. (4) this intercept is increased by the material constant A. Figure 7 shows

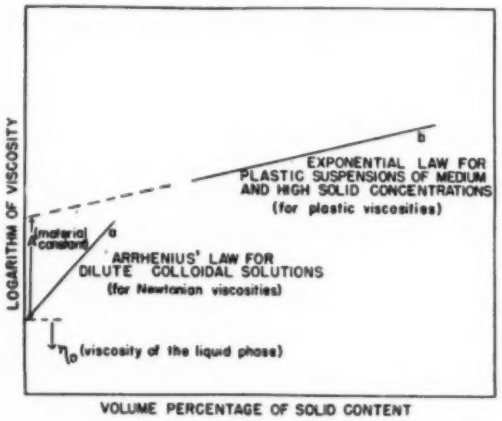

FIG. 7. Schematic drawing of Arrhenius' exponential law for Newtonian viscosities of dilute suspensions, curve a, and of the exponential law established for plastic viscosities of plastic suspensions of high solid concentration, curve b.

schematically the difference between the two laws. Since at zero percent of solid content the viscosity of the suspension must be equal to the viscosity of the liquid phase, which in case of the pigment suspensions tested is the viscosity of the oil, Eq. (4) is correct only if A also becomes zero at zero percent solid content. This means, as seen from Fig. 7, that either the straight line b has to curve towards the percentage axis; or, that the line b changes into line a at a certain low percent solid content, following Arrhenius' law for the lower solid percentages, where the suspensions are dilute enough to flow like Newtonian liquids. The change from one exponential law to the other may be explained, if one considers that the suspensions are Newtonian liquids up to a certain solid content, and that at a higher percent of solid present the suspensions change fairly suddenly to plastic materials.

Yield Value as Function of Concentration

The interesting fact is that not only the plastic viscosities of pigment suspensions, but also the yield values for suspensions of higher pigment content are logarithmically related to the volume percent concentration. At low concentrations (below 25 percent) the yield values of two pigment suspensions do not conform to the exponential law, Fig. 8, indicating that at low percent solid content the straight lines curve towards the percent axis. Such a deviation from the straight line at low solid concentration is logically expected, since at zero percent concentration the oil alone is left, which does not exhibit any yield value at the rate of shear at which these experiments have been carried out. Because of the curvature at low concentrations, it is possible that the straight line obtained at higher concentrations is also a part of the curve, but is curved so slightly that it approximates a straight line for all practical purposes.

In Figs. 8 and 9 the logarithm of the yield value is plotted against the volume percent of solid content for the same pigment suspensions as are shown in Figs. 4 and 5, respectively. The easily deduced equation is:

$$f = Me^{Np} \tag{5}$$

where M and N, see Table II, are two constants depending on the material. Just like the constants A and B, they probably are related to the particle size, particle shape, and to the top rate of shear at which the material has been measured. They will also depend on the wetting characteristic of pigment and liquid. Correlating the constant N (the slope) with the average particle diameter, d_3, Fig. 6 (Table III), a logarithmic relationship is evolved, indicating that N, among other factors, depends on the particle size.

Extensions of Experimental Investigations to Oils

In a recent paper[12] it has been shown that oils, if subjected to a high rate of shear beyond a certain rate of shear which has been called the "limiting rate of shear," behave like thixotropic

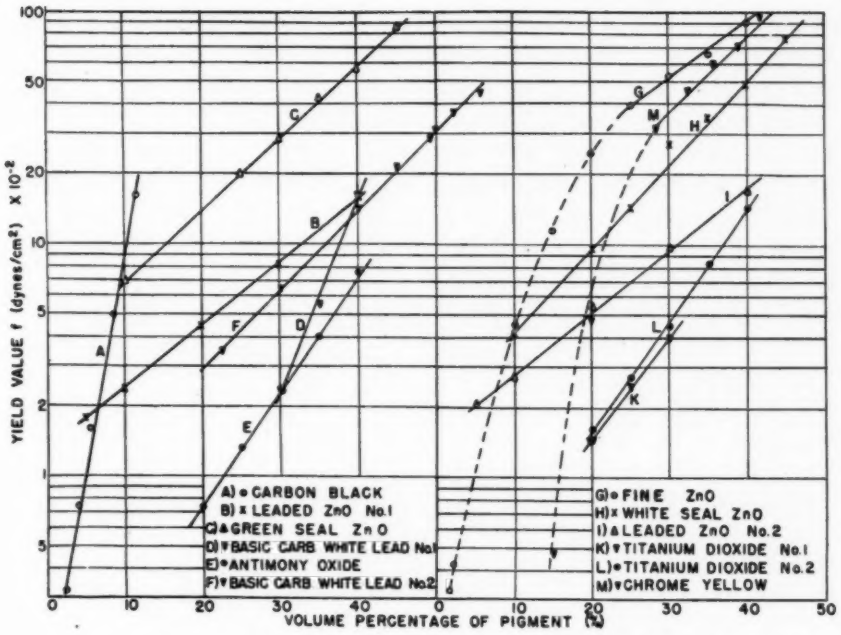
YIELD VALUE f (dynes/cm²) X 10⁻²
VOLUME PERCENTAGE OF PIGMENT (%)
A) • CARBON BLACK
B) × LEADED ZnO No.1
C) ▲ GREEN SEAL ZnO
D) ▼ BASIC CARB WHITE LEAD No.1
E) • ANTIMONY OXIDE
F) ▼ BASIC CARB WHITE LEAD No.2
G) • FINE ZnO
H) × WHITE SEAL ZnO
I) ▲ LEADED ZnO No.2
K) ▼ TITANIUM DIOXIDE No.1
L) • TITANIUM DIOXIDE No.2
M) ▼ CHROME YELLOW

FIG. 8. The logarithm of yield value as a function of solid concentration by volume for various pigment suspensions.

plastics, while below that rate of shear they behave like Newtonian liquids. Though Gussman[18] and Mikusch[19] have satisfactorily proven that the viscosity of mixtures of linseed and dehydrated castor oils follows an exponential law similar to Arrhenius' law for colloidal solutions and suspensions, it was not clear whether this exponential law for oil mixtures would apply in the region above the limiting rates of shear where the oils show thixotropic and plastic characteristics. The authors of this paper therefore extended their investigation to the plastic region of oil mixtures by employing the same experimental method which was used for suspensions and solutions. A series of measurements was made using various top rates of shear as parameters. Figure 10 shows the results of these experiments. It will be noticed that if the top rate of shear is exceedingly low, a straight line is obtained for all different oil-oil ratios. This, however, is the case only if the top rate of shear is below the limiting rate of shear, even for the heavier oil contained in the oil mixture. If the top rate of shear is so high that it exceeds the limiting rate of shear for any one concentration of the oil mixture, then a deviation from the straight line must be logically expected and has been experimentally found. Figure 10 shows a representative diagram of the change in viscosity of a mixture of two isobutylene oils of different Newtonian viscosity when subjected to various constant top rates of shear.

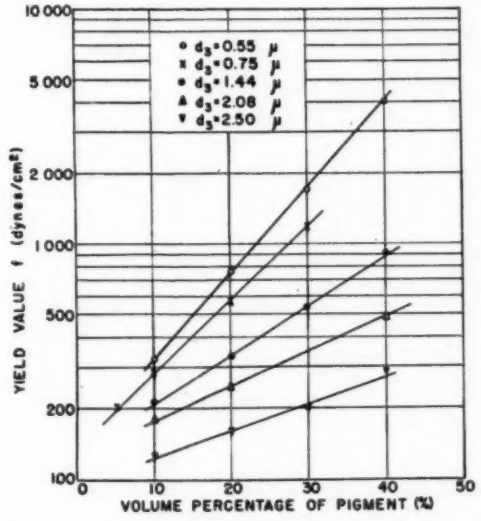

FIG. 9. The logarithm of yield value as a function of solid concentration by volume for various suspensions of a leaded zinc oxide of varying average particle diameter d_3.

[18] Gussman, Am. Paint J. Convention Daily 16, 18, 20 (1939).
[19] Von Mikusch, Ind. Eng. Chem. Ind. Ed. **32**, 1061 (1940).

It should be recalled that Gussman and Mikusch investigated oil mixtures containing only two oils of the same type. The question therefore arose whether the same laws also held for oil mixtures with constituents of different types. The authors tested oil mixtures containing linseed, mineral, and isobutylene oils within the Newtonian region as well as above the Newtonian region. In Fig. 11 the results are shown for mixtures tested within the Newtonian region. It will be seen that a law similar to the one already established by Arrhenius for Newtonian viscosities of colloidal solutions is correct for the Newtonian viscosities of oil mixtures with same and with different type constituents, if the oils are miscible. The equation deduced from Fig. 11 is:

$$\eta = \eta_1 e^{[p \ln (\eta_2/\eta_1)]/100} \qquad (6)$$

where η is the Newtonian viscosity of the mixture, η_1 and η_2 are the Newtonian viscosities of the two oils, and p is the volume percentage at which the heavier oil is contained in the mixtures. Regarding measurements above the Newtonian region, Fig. 10, as already previously described, is representative.

CONCLUSION

In the paint and printing ink industries in which the authors have specialized, it has proved helpful to know that an exponential law, similar to the one originally conceived by Arrhenius for Newtonian viscosities of colloidal suspensions is essentially correct for plastic viscosities of plastic materials and also for yield values in a region from medium to high concentration. If two suspensions of different but fairly high concentration are available, the application of this law makes it possible to prepare any other concentration with a predictable plastic viscosity and yield value. Instead of a trial and error method, which is always time consuming, any desired mixture can be obtained on the basis of approximate calculations from the exponential equations given in this paper.

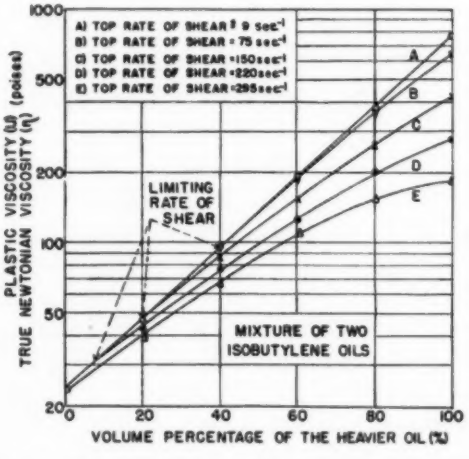
A) TOP RATE OF SHEAR = 9 sec⁻¹
B) TOP RATE OF SHEAR = 75 sec⁻¹
C) TOP RATE OF SHEAR = 150 sec⁻¹
D) TOP RATE OF SHEAR = 220 sec⁻¹
E) TOP RATE OF SHEAR = 295 sec⁻¹
PLASTIC VISCOSITY (η) (poises)
TRUE NEWTONIAN VISCOSITY (η)
LIMITING RATE OF SHEAR
MIXTURE OF TWO ISOBUTYLENE OILS
VOLUME PERCENTAGE OF THE HEAVIER OIL (%)

FIG. 10. The logarithm of Newtonian and plastic viscosity of a mixture of two isobutylene oils of different viscosity, obtained at various constant top rates of shear, as a function of the volume percent content of the heavier oil.

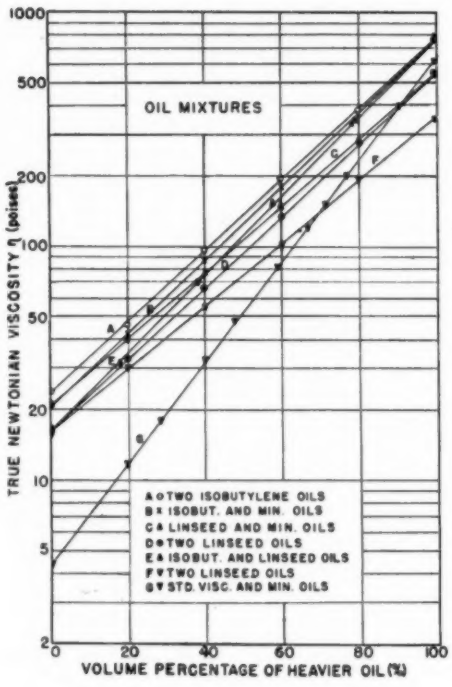
OIL MIXTURES
TRUE NEWTONIAN VISCOSITY η (poises)
A ○ TWO ISOBUTYLENE OILS
B × ISOBUT. AND MIN. OILS
C ▲ LINSEED AND MIN. OILS
D ● TWO LINSEED OILS
E ▲ ISOBUT. AND LINSEED OILS
F ▼ TWO LINSEED OILS
G ▼ STD. VISC. AND MIN. OILS
VOLUME PERCENTAGE OF HEAVIER OIL (%)

FIG. 11. The logarithm of the Newtonian viscosity of various oil mixtures as a function of the volume percent content of the heavier oil.

ACKNOWLEDGMENT

The authors are grateful to Interchemical Corporation for the permission to publish this paper, to Evelyn Berezin for assistance in obtaining the experimental data, and to Ernest F. Fullam for the measurements of the average particle diameters, d_3.

Transmission-Line Theory and Its Application

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In Part I several forms of the solution of the transmission-line equations are discussed including, in particular, a completely hyperbolic one in which the over-all attenuation and phase shift of the terminal impedances are expressed in a form involving terminal functions which simplify the analysis of many complex problems. Formulas for the input resistance, reactance, conductance, and susceptance of a section of line of any length and terminated in an arbitrary impedance are given in completely general terms. Curves computed from these formulas for four typical terminations are shown. A general circle diagram for determining the input impedance and admittance as well as the terminal functions is described. It consists of circles of constant over-all attenuation and circles of constant over-all phase shift. The distribution of current is represented in general terms suitable for a line driven at one end or by a generator coupled loosely at any point along the line. In Part II the general formulas described in Part I are applied to practical problems. A simple and completely general formula is derived for determining the transfer of power and the efficiency of transmission for a line of any length terminated in an arbitrary load. An equally simple and general formula for the standing wave ratio is given. Application of the formulas for input impedance and admittance to the problem of bead spacing on resonant and nonresonant lines and to series and shunt impedance transforming or matching sections is outlined. The experimental determination of attenuation and phase constants and of "Q" for the line and for the terminations is discussed. The application of formulas and experimental methods to include hollow pipe transmission lines or wave guides is considered briefly in Part III. A method for defining and measuring terminating impedances for hollow pipes is described.

INTRODUCTION

THE well-known differential equations,

$$-\partial V/\partial z=(r+j\omega l)I, \tag{1a}$$

$$-\partial I/\partial z=(g+j\omega c)V, \tag{1b}$$

are fundamental in the analysis of all transmission lines whether of the open-wire, the coaxial, or the hollow metal pipe (wave-guide) type.

For open-wire and coaxial lines which are very closely spaced compared with the wave-length, V and I are, respectively, the scalar potential difference between the two conductors and the total current in one of these. It is assumed that currents in the two conductors are equal and opposite. If this is not true the unequal currents can always be separated analytically into a pair of equal and opposite currents plus a pair of equal currents in the same direction. The Eqs. (1) apply to the former currents only; the latter constitute "antenna" currents and must be treated in terms of antenna theory.[1] For hollow metal pipes and coaxial lines which are of sufficient cross-sectional size to permit higher modes in the distribution of current and charge, V and I are potential and stream functions from which the densities of current and charge or the electromagnetic field may be computed. The present discussion is not concerned with the approximations involved in the derivation of (1) for any of the several types of transmission lines, but rather with the form and application of the solution from the point of view of convenience in interpretation and in practical application. The discussion will be carried through first from the point of view of conventional two-conductor transmission lines. Application to hollow metal pipes will be made later.

In standard treatments of transmission lines the solution of (1) is usually expressed in one of two forms which will be called the exponential form and the conventional hyperbolic form, respectively. They will be outlined below. A third form of the solution which offers many advantages will then be described. This will be called the completely hyperbolic form. All three forms of the solution are expressed in terms of the four parameters of the line, r, l, c, and g as combined in the complex propagation constant k and the characteristic impedance Z_c of the line. These are defined below.

$$k=\alpha+j\beta=[(r+j\omega l)(g+j\omega c)]^{\frac{1}{2}}, \tag{2}$$

$$Z_c=R_c(1-j\phi)=[(r+j\omega l)/(g+j\omega c)]^{\frac{1}{2}}. \tag{3}$$

[1] This is discussed in some detail in R. King, Proc. I.R.E. 31 (Nov. 1943).

Here α is the attenuation constant in nepers per meter, β the phase constant in radians per meter, R_c is the characteristic resistance in ohms, and ϕ is the dimensionless distortion factor. Explicit formulas for the general case are given in reference 4, Eqs. (8)–(15), and for the case of low attenuation in Eqs. (65). It is well to note that for all types of lines which are constructed using good conductors and dielectrics* α/β and ϕ are sensibly equal and small compared with unity (this is true only in the pass band for hollow pipes). For many purposes α/β and ϕ may be neglected; for others they may not.

PART I: OUTLINE OF THE THEORY

The Exponential Solution

The exponential form of the solution of (1) for the circuit of Fig. 1 is found in standard treatises.

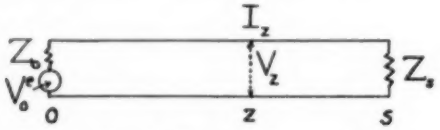

FIG. 1. Circuit diagram of two-wire line.

It may be summarized as follows:

$$I_z=\frac{V_0^e}{Z_0+Z_c}\left[\frac{e^{-kz}+\Gamma_s e^{-k(2s-z)}}{1-\Gamma_0\Gamma_s e^{-2ks}}\right], \tag{4a}$$

$$V_z=\frac{V_0^e Z_c}{Z_0+Z_c}\left[\frac{e^{-kz}-\Gamma_s e^{-k(2s-z)}}{1-\Gamma_0\Gamma_s e^{-2ks}}\right]. \tag{4b}$$

Here

$$\Gamma_0=(Z_c-Z_0)/(Z_c+Z_0), \tag{4c}$$

$$\Gamma_s=(Z_c-Z_s)/(Z_c+Z_s). \tag{4d}$$

The input impedance of a section of line of length s is obtained directly from (4a) and (4b). Thus

$$V_0/I_0=Z_{in}=Z_c[(1-\Gamma_s e^{-2ks})/(1+\Gamma_s e^{-2ks})]. \tag{4e}$$

The principal advantage of the exponential solution lies in the physical interpretations which may be attached to it. Thus if the denominator in the bracket of (4b) is divided into the numerator one obtains an infinite series which suggests the interpretation that the current I_z at any point z along the line is the resultant of contributions due to disturbances which have traveled back and forth along the line experiencing reflections at each termination. The complex coefficient of reflection Γ_0 (or Γ_s) is interpreted to describe the effect on the amplitude and phase of such a disturbance at *one* reflection at Z_0 (or Z_s). Frequently this infinite train of disturbances as represented by the infinite series is considered in two parts corresponding to the series obtained by dividing, respectively and separately, the first and second terms in the numerator by the denominator. The one part is visualized to represent a composite disturbance traveling in one direction, the other part a similarly composite but different disturbance traveling in the other direction along the line. These are called direct and reflected waves. From the mathematical point of view the exponential form of the solution is intricate and cumbersome except in one academic and one practical special case in which it is exceedingly simple and useful. These are the cases of the infinite and the so-called non-resonant lines defined respectively by:

$$s=\infty, \tag{5a}$$

$$Z_s=Z_c. \tag{5b}$$

In both cases (4a) and (4b) reduce to the same simple form:

$$I_zZ_c=V_z=\left(\frac{V_0^eZ_c}{Z_0+Z_c}\right)e^{-kz}. \tag{5c}$$

The Conventional Hyperbolic Solution

If complex hyperbolic functions are used in the solution of (1) instead of exponentials while the terminal impedances are written explicitly, the following form of the solution is obtained.

$$I_z=V_0^e\left[\frac{Z_s\sinh kw+Z_c\cosh kw}{(Z_c^2+Z_0Z_s)\sinh ks+Z_c(Z_0+Z_s)\cosh ks}\right], \tag{6a}$$

$$V_z=V_0^eZ_c\left[\frac{Z_s\cosh kw+Z_c\sinh kw}{(Z_c^2+Z_0Z_s)\sinh ks+Z_c(Z_0+Z_s)\cosh ks}\right]. \tag{6b}$$

*Specifically $g/c\ll r/l$, $(r/wl)^2\ll 1$.

Here w is the distance from the load impedance Z_s, to the point z where I_z and V_z are computed. Thus

$$w = s - z. \tag{6c}$$

The input impedance of a line of length s is:

$$Z_{in} = Z_c \left\{ \frac{Z_s \cosh ks + Z_c \sinh ks}{Z_s \sinh ks + Z_c \cosh ks} \right\}. \tag{6d}$$

An alternative equivalent form expressed in terms of the current I_s entering, and the potential difference V_s across Z_s is:

$$I_z = (V_s/Z_c) \sinh kw + I_s \cosh kw, \tag{7a}$$

$$V_z = V_s \cosh kw + I_s Z_c \sinh kw. \tag{7b}$$

The conventional hyperbolic solution reduces to simple and useful forms when Z_s is either infinite or zero. It does not lend itself to physical interpretation; and in the general case it is mathematically complex and unattractive.

The Completely Hyperbolic Solution

The completely hyperbolic form differs from the conventional hyperbolic one only in going one step further and expressing the terminal impedances in terms of hyperbolic functions. If one introduces the following notation defining the complex terminal functions (or hyperbolic angles of the terminal impedances as they were called by Kennelly), $\theta_0 = \rho_0 + j\Phi_0$, $\theta_s = \rho_s + j\Phi_s$, for the terminating impedances Z_0 and Z_s, respectively,

$$Z_0 = Z_c \coth \theta_0; \quad Z_s = Z_c \coth \theta_s, \tag{8a}$$

and substitutes these in (6a, b) one obtains

$$I_z = \frac{V_0^e}{Z_c} \left[\frac{\sinh \theta_0 \sinh (kw + \theta_0)}{\sinh (ks + \theta_0 + \theta_s)} \right], \tag{8b}$$

$$V_z = V_0^e \left[\frac{\sinh \theta_0 \cosh (kw + \theta_s)}{\sinh (ks + \theta_0 + \theta_s)} \right]. \tag{8c}$$

With the notation (8a) the input impedance of a terminated section of length s as given by (6d) becomes:

$$Z_{in} = Z_c \coth (ks + \theta_s). \tag{8d}$$

The principal advantage of the completely hyperbolic form of the solution lies in the significant fact that the *over-all* effect of each termination (as distinct from its effect at a single reflection which is described by the coefficients of reflection Γ_0 and Γ_s) is expressed in terms of the functions $\theta_0 = \rho_0 + j\Phi_0$ and $\theta_s = \rho_s + j\Phi_s$ in which the ρ's describe the over-all attenuation, the Φ's the complete phase shift of the terminations in such a way that they may be added, respectively and directly, to the over-all attenuation αs, and the over-all phase shift βs, of the line itself. The general case involving perfectly arbitrary terminations is thus reduced to the analytical simplicity of the special cases with $Z_s = 0$ or $Z_s = \infty$ in the conventional hyperbolic form. Thus the completely hyperbolic solution does not make special cases which are simple in the exponential or conventional hyperbolic form simpler; it makes the general case, which is excessively intricate in both of the other forms, practically as simple as the special cases. This will become apparent below where the input impedance, the transfer of power, the standing wave ratio, and matching sections are treated.

The Input Impedance and the Terminal Functions

The defining relations (8a) and (8d) may be expanded into

$$Z_s = Z_c \coth [\rho_s + j\Phi_s], \tag{9}$$

$$Z_{in} = Z_c \coth [(\alpha s + \rho_s) + j(\beta s + \Phi_s)]. \tag{10}$$

Expressions (9) and (10) are formally alike. By writing $s=0$, (10) reduces to (9). For convenience in writing let

$$A_s = \alpha s + \rho_s, \tag{11}$$

$$F_s = \beta s + \Phi_s, \tag{12}$$

so that A_s is the over-all attenuation, F_s the over-all phase shift of a section of line of length s and characteristic impedance Z_c, and with termination Z_s. If $s=0$, then A_s reduces to ρ_s, F_s to Φ_s. In this latter case if the subscript s is changed to 0, the analysis applies to Z_0. Thus the formula

$$Z_{in} = Z_c \coth (A_s + jF_s) \tag{13}$$

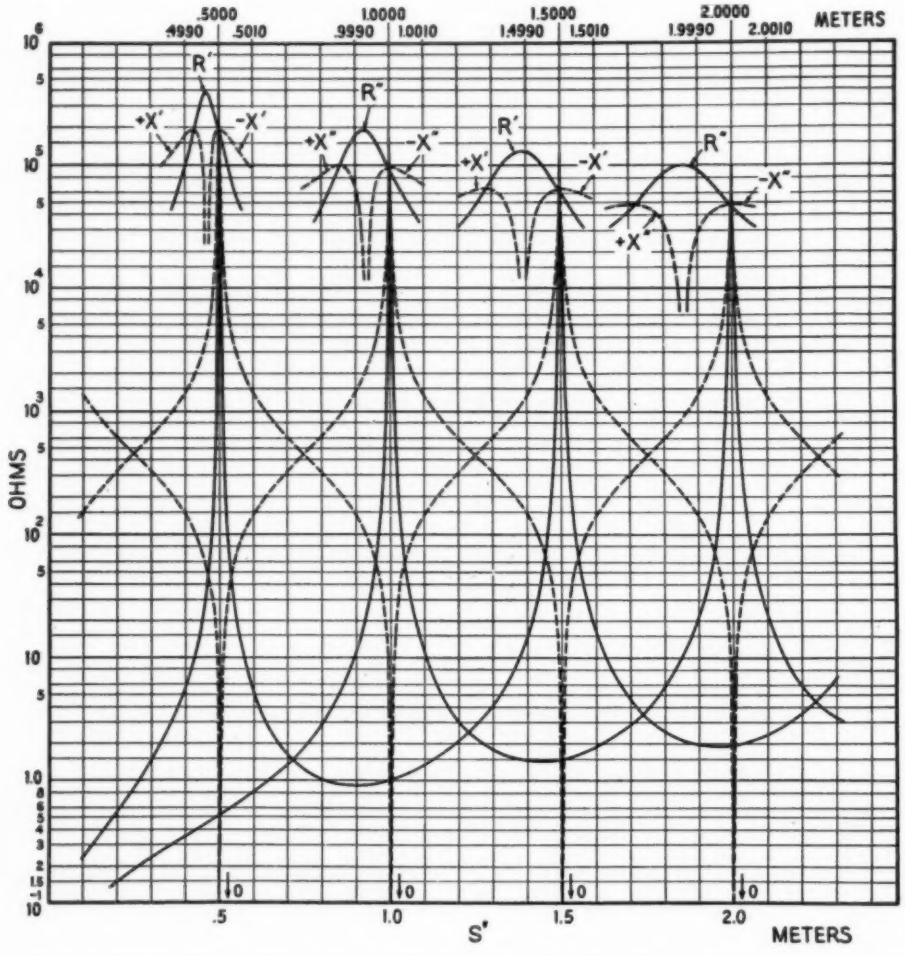

FIG. 2. Input impedance of section of two-wire line of length $s=s'$ for an open end, $s=s'-k_s$ for a bridged end. $R_c=439.8$ ohms, $\phi=7.184\times10^{-4}$; $\alpha=2.258\times10^{-3}$ neper/m, $\beta=3.144$ radians/m; wire radius $a=5.118\times10^{-2}$ cm; spacing $b=2$ cm; $\lambda_{air}=2$ m; $Z_{in}'=R_{in}'+jX_{in}'$ for bridged end (same wire); $Z_{in}''=R_{in}''+jX_{in}''$ for open end; $k_s\doteq b/2=1$ cm.

may stand for (9) or (10) with appropriate choice of symbols and subscript. This expression is readily solved for R_{in} and X_{in}. Thus:

$$R_{in}=R_c\left[\frac{\sinh 2A_s-\phi\sin 2F_s}{\cosh 2A_s-\cos 2F_s}\right], \tag{14}$$

$$X_{in}=-R_c\left[\frac{\sin 2F_s+\phi\sinh 2A_s}{\cosh 2A_s-\cos 2F_s}\right]. \tag{15}$$

By writing ρ_s for A_s, Φ_s for F_s, R_s for R_{in}, and X_s for X_{in} the above formulas define the terminal resistance and reactance in terms of the attenuation and phase functions. Over most of the range of βs the small term which has ϕ as a factor is negligible. The input resistance and reactance of any terminated section of line may be computed directly from these formulas. Curves for R_{in} and X_{in} for

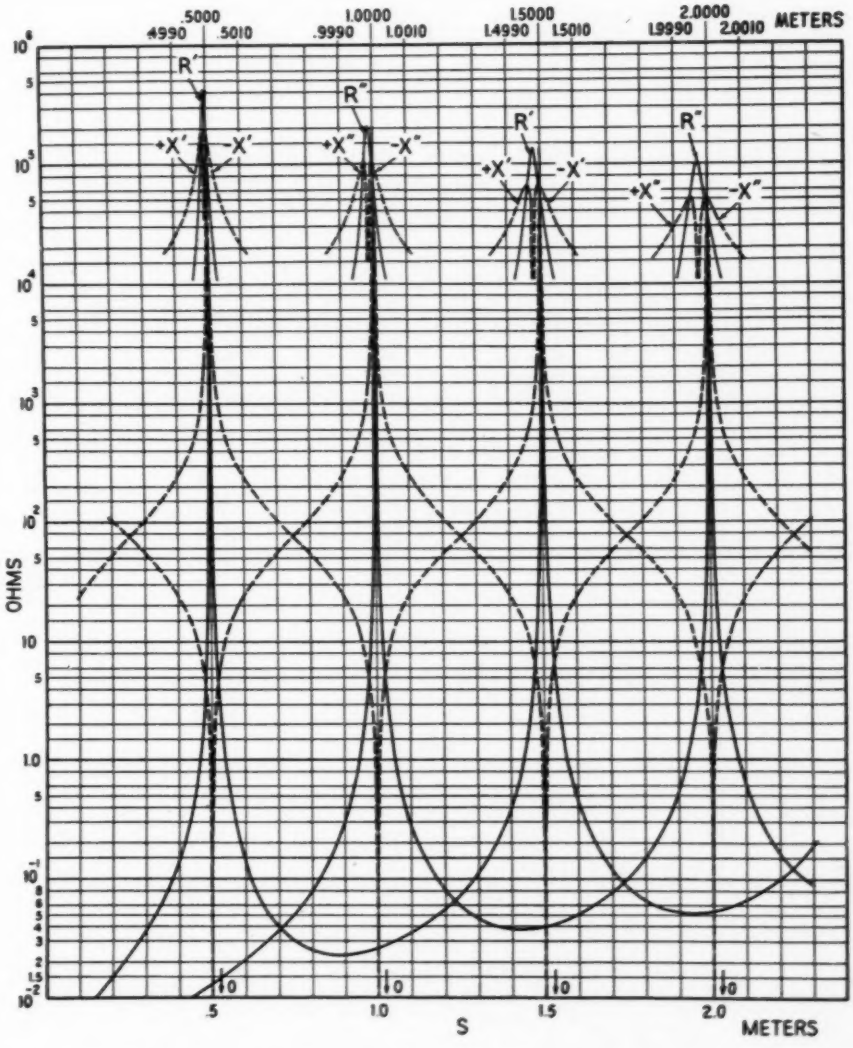

FIG. 3. Input impedance of a section of coaxial line of length s. $R_c=75.13$ ohms, $\phi=1.384\times10^{-4}$, $\alpha=3.577\times10^{-4}$ neper/m, $\beta=3.142$ radians/m; radius of inner conductor $a=1$ cm, inner radius of outer conductor $b=3.5$ cm; $\lambda_{air}=2$ m; $Z_{in}'=R_{in}'+jX_{in}'$ for closed end (piston); $Z_{in}''=R_{in}''+jX_{in}''$ for open end.

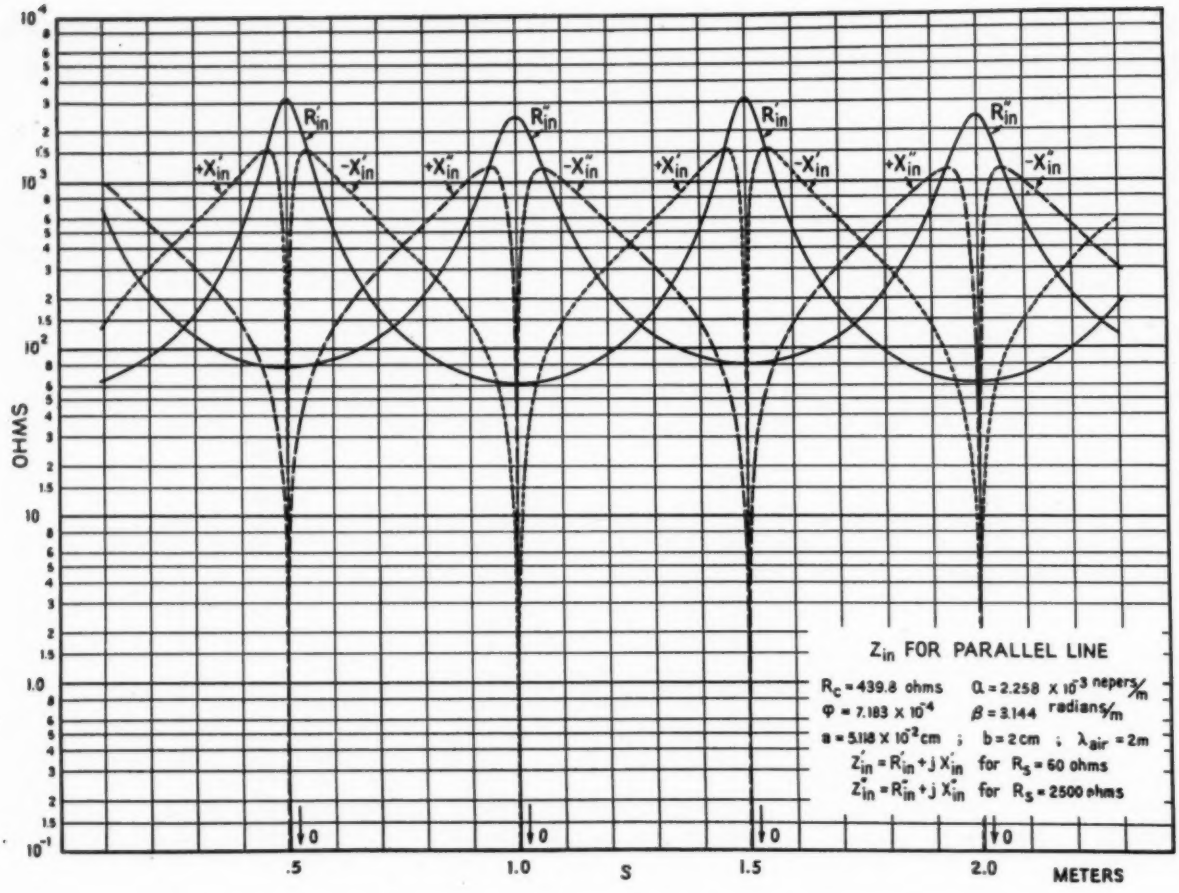

FIG. 4. Input impedance of the same line used in Fig. 2 but with different terminations.

$Z_{in}' = R_{in}' + jX_{in}'$ for $R_s = 60$ ohms; $X_s = 0$.
$Z_{in}'' = R_{in}'' + jX_{in}''$ for $R_s = 2500$ ohms; $X_s = 0$.

a parallel wire line and a coaxial line over a wide range of lengths s as computed from (14) and (15) are shown in Figs. 2–5 for four different terminal impedances.

Formulas (14) and (15) may be solved for A_s and F_s. One obtains:

$$A_s = \tfrac{1}{2} \tanh^{-1}\left[\frac{2R_c(R_{in} - \phi X_{in})}{(R_{in}^2 + X_{in}^2) + R_c^2(1+\phi^2)}\right], \tag{16}$$

$$F_s = \tfrac{1}{2} \tan^{-1}\left[\frac{-2R_c(X_{in} + \phi R_{in})}{(R_{in}^2 + X_{in}^2) - R_c^2(1+\phi^2)}\right]. \tag{17}$$

By writing ρ_s for A_s, Φ_s for F_s and changing the subscript, in, to s the above expressions may be used for computing ρ_s and Φ_s in terms of Z_s and Z_c. For all good lines ϕ is negligible compared with unity so that over most of the ranges of R_{in} and X_{in} the terms in ϕ may be neglected.

For many purposes it is convenient to define the dimensionless ratios r_1 and x_1, by writing

$$Z_{in}/Z_c = r_{1in} + jx_{1in}, \tag{18}$$

where

$$r_{1in}=\frac{R_{in}-\phi X_{in}}{R_c(1+\phi^2)}=\frac{\sinh 2A_s}{\cosh 2A_s-\cos 2F_s}, \tag{19a}$$

$$x_{1in}=\frac{X_{in}+\phi R_{in}}{R_c(1+\phi^2)}=\frac{-\sin 2F_s}{\cosh 2A_s-\cos 2F_s}. \tag{19b}$$

Except for special cases the terms in ϕ are negligible with good lines so that for most purposes

$$r_{1in}\doteq R_{in}/R_c; \quad x_{1in}\doteq X_{in}/R_c. \tag{20}$$

If (19a) and (19b) are solved for A_s and F_s one obtains:

$$A_s=\tfrac{1}{2}\tanh^{-1}\left[\frac{2r_{1in}}{r_{1in}^2+x_{1in}^2+1}\right], \tag{21}$$

$$F_s=\tfrac{1}{2}\tan^{-1}\left[\frac{-2x_{1in}}{r_{1in}^2+x_{1in}^2-1}\right]. \tag{22}$$

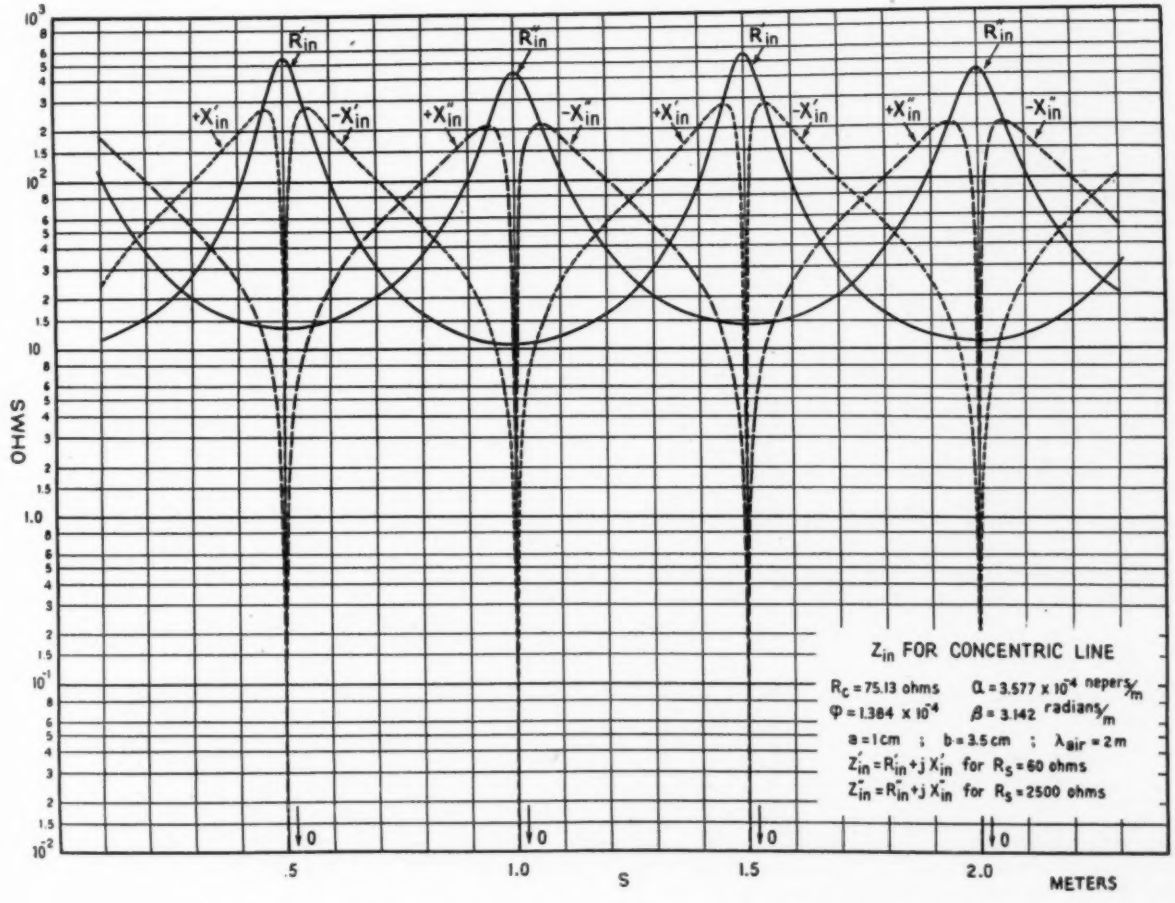

FIG. 5. Input impedance of the same line used in Fig. 3 but with different terminations.
$Z_{in}'=R_{in}'+jX_{in}'$ for $R_s=60$ ohms; $X_s=0$.
$Z_{in}''=R_{in}''+jX_{in}''$ for $R_s=2500$ ohms, $X_s=0$.

By changing the subscript, *in*, to *s* and writing ρ_s for A_s, Φ_s for F_s, formulas (18)–(22) apply to the terminal impedance Z_s and the terminal attenuation and phase functions, ρ_s and Φ_s. The octant of the function Φ_s or F_s may be obtained from Fig. 6.

It is readily verified that curves of constant A_s and curves of constant F_s are families of circles in the r_1, x_1 plane. The significant characteristics of these circles are given in Table I. A circle diagram is shown in Fig. 7. Circles of constant A_s are indicated by the numerical value of A_s in nepers; circles of constant F_s are designated with numerical values of F_s in degrees.

TABLE I.

Characteristic	Circles of constant A_s	Circles of constant F_s
Coordinates of center	$\begin{cases} r_1=\coth 2A_s \\ x_1=0 \end{cases}$	$\begin{cases} r_1=0 \\ x_1=-\cot 2\Phi_s \end{cases}$
Radius of circle	$\operatorname{csch}^2 2A_s$	$\csc^2 2F_s$
Intercepts with coordinate axes	$\begin{cases} r_1=\tanh A_s; & r_1=\coth A_s \\ x_1=0; & x_1=0 \end{cases}$	$\begin{cases} r_1=0; & r_1=0 \\ x_1=\tan F_s; & x_1=-\cot F_s \end{cases}$

The input impedance of a section of line of length s terminated in Z_s is determined very quickly using the circle diagram. By using given values of R_s, X_s, for the termination and R_c and ϕ for the line, r_{1s} and x_{1s} are readily computed using the left sides of (19a) and (19b), or (20) with appropriate change in subscript from *in* to *s*. Using these values of r_{1s} and x_{1s} to locate a point in the circle diagram, the values of ρ_s and Φ_s which locate the same point are at once determined. By adding αs to ρ_s, A_s is next determined; by adding βs to Φ_s, F_s is obtained. Using these values of A_s and F_s to locate a point in the circle diagram, the corresponding values of r_{1in} and x_{1in} are read from the diagram. R_{in} and X_{in} are then readily computed from (19a) and (19b) solved for R_{in} and X_{in}, *viz.*:

$$R_{in}=R_c(r_{1in}+\phi x_{1in}), \tag{23}$$

$$X_{in}=R_c(x_{1in}-\phi r_{1in}). \tag{24}$$

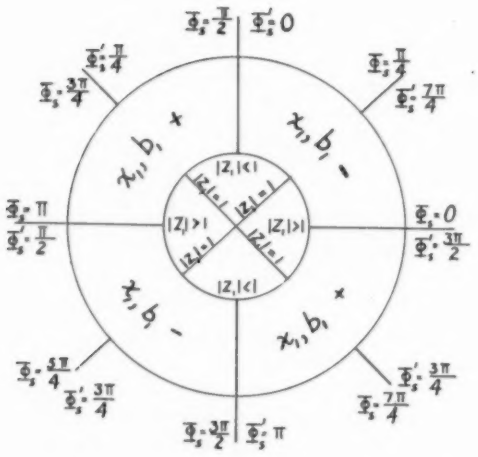

FIG. 6. Quadrant and octant representation of the phase function Φ_s for a termination Z_s in terms of the sign of the ratio factor x_1 and the magnitude $|z_1|=(r_1^2+x_1^2)^{\frac{1}{2}}$. Similarly for Φ_s' $(=\Phi-\pi/2)$ in terms of the sign of b_1 and the magnitude of $|y_1|=1/|z_1|$. If F_s is written for Φ_s throughout, the diagram applies to the phase factor for a terminated section of line.

In most cases, the simple formulas (20) are adequate.

Critical Values of the Input Impedance

Relatively simple expressions for important critical values of the input impedance may be written for a line with low over-all attenuation, low attenuation per unit length, and negligible leakage conductance. They are summarized below.

Conditions assumed:

$$A_s^2 \ll 1, \tag{25}$$

$$\phi^2=(\alpha/\beta)^2 \ll 1. \tag{26}$$

Simple formulas in a restricted range:

$$R_{in}=R_c[A_s \csc^2 F_s-\phi \cot F_s], \tag{27}$$

$$X_{in}=-R_c \cot F_s, \tag{28}$$

provided, in this range,

$$\sin^2 F_s \gg A_s^2. \tag{29}$$

Conditions for input resonance and anti-resonance:

Input resonance:

$$X_{in}=0; \quad R_{in} \text{ near minimum (except for } n=0), \tag{30a}$$

$$\text{at } \beta s=(2n+1)(\pi/2)-\Phi_s. \tag{30b}$$

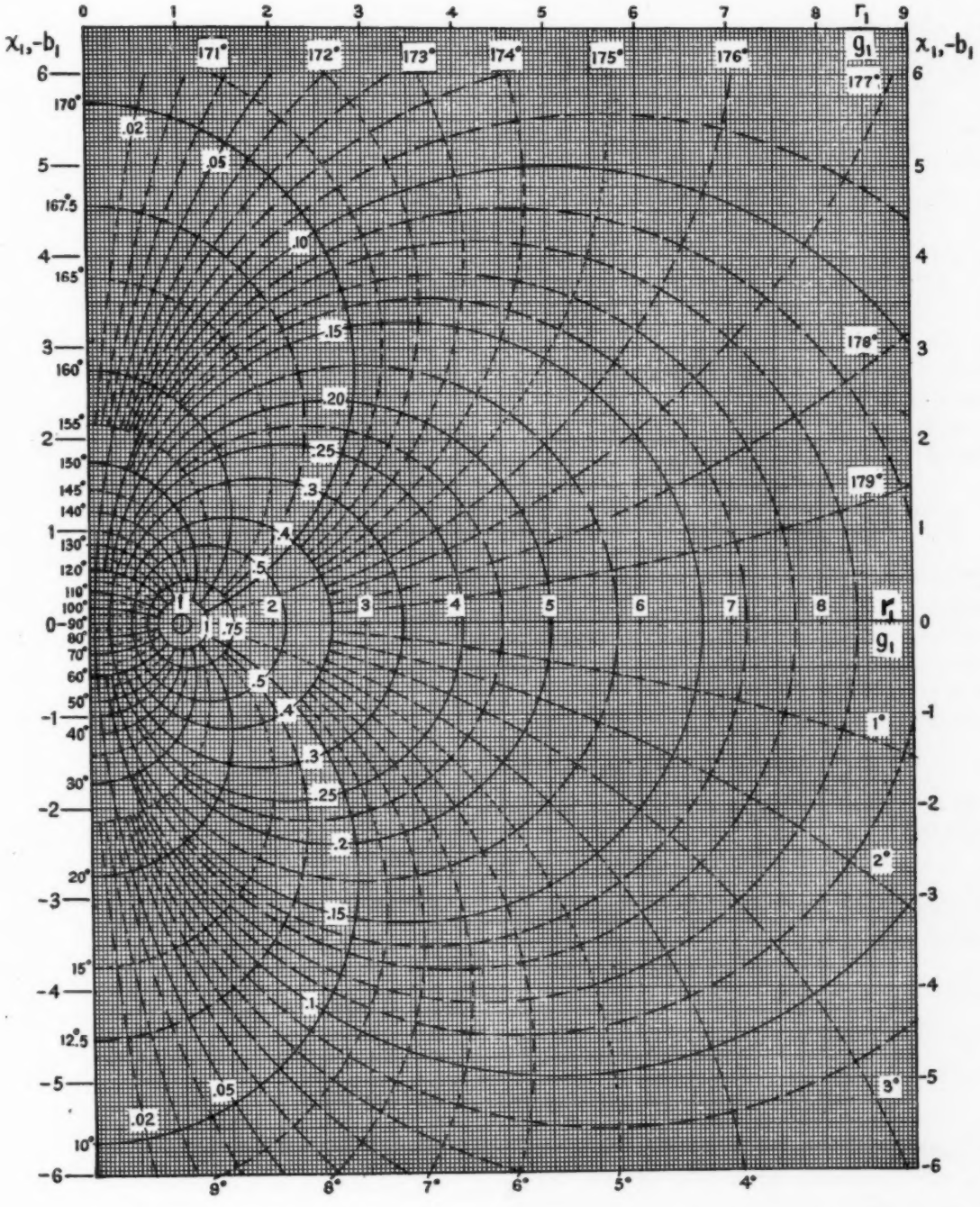

FIG. 7. Circle diagram showing circles of constant attenuation (circles marked in nepers from 0.02 to 1) and constant phase (circles marked in degrees from 0° to 179°). With r_1, x_1 as coordinates the circles of constant phase give F_s or Φ_s; with g_1 and $-b_1$ as coordinates the circles of constant phase give F_s' or Φ_s'.

Input anti-resonance:

$$X_{in}=0; \quad R_{in} \text{ at maximum}, \tag{31a}$$

$$\text{at } \beta s=n\pi-\Phi_s. \tag{31b}$$

Extreme values:

$$(R_{in})_{\min}=R_cA_s, \tag{32a}$$

$$\text{at } \tan(\beta s+\Phi_s)=\beta(s+\rho_s/\alpha). \tag{32b}$$

$$(R_{in})_{\max}=R_c/A_s, \tag{33a}$$

$$\text{at } \beta s=n\pi-\Phi_s. \tag{33b}$$

$$(X_{in})_{\max}=(R_c/2A_s)(1-\alpha/\beta), \tag{34a}$$

$$\text{at } \beta s=(n\pi-\Phi_s)(1-\alpha/\beta)-\rho_s. \tag{34b}$$

$$(X_{in})_{\min}=-(R_c/2A_s)(1+\alpha/\beta), \tag{35a}$$

$$\text{at } \beta s=(n\pi-\Phi_s)(1+\alpha/\beta)+\rho_s. \tag{35b}$$

These formulas are useful in designing tank circuits for u.h.f. oscillators, in constructing insulating stubs and switching arrangements using sections of transmission line.

The Input Admittance and the Terminal Functions

In matching problems terminated sections of transmission line are connected in parallel, so that expressions for the input susceptance and conductance are more convenient than formulas for input reactance and resistance. They are very easily obtained. From (8d) one has:

$$Y_{in}=1/Z_{in}=(1/Z_c)\tanh(ks+\theta_s). \tag{36}$$

Since it is readily verified by using standard formulas that

$$\coth(x+jy)=\tanh(x+jy\pm j\pi/2), \tag{37}$$

an alternative form is possible and actually more convenient. Thus

$$Y_{in}=(1/Z_c)\coth(ks+\theta_s'), \tag{38}$$

where

$$\theta_s'=\rho_s+j\Phi_s'=\theta_s\pm j\pi/2. \tag{39a}$$

Hence, choosing the negative sign,

$$\Phi_s'=\Phi_s-\pi/2. \tag{39b}$$

The expression (38) for Y_{in} differs from (8d) for Z_{in} only in containing θ_s' instead of θ_s, and having Z_c in the denominator instead of in the numerator. Since

$$1/Z_c=(1+j\phi)/R_c(1+\phi^2), \tag{40}$$

the real and imaginary parts of Y_{in} may be obtained from those of Z_{in} by writing Φ_s' for Φ_s, dividing by $R_c^2(1+\phi^2)$, and changing the sign of ϕ. Thus with

$$Y_{in}=G_{in}-jB_{in} \tag{41}$$

one has from (14) and (15)

$$G_{in}=\frac{1}{R_c(1+\phi^2)}\left[\frac{\sinh 2A_s+\phi\sin 2F_s'}{\cosh 2A_s-\cos 2F_s'}\right], \tag{42}$$

$$B_{in}=\frac{1}{R_c(1+\phi^2)}\left[\frac{\sin 2F_s'-\phi\sinh 2A_s}{\cosh 2A_s-\cos 2F_s'}\right]. \tag{43}$$

Here

$$F_s'=\beta s+\Phi_s'=\beta s+\Phi_s\pm\pi/2. \tag{44}$$

Except for the small term in ϕ, a constant factor, and the appearance of F_s' instead of F_s the expressions for G_{in} and B_{in} are like those for R_{in} and X_{in}. Accordingly the curves for G_{in} and B_{in} are essentially the same in shape as those for R_{in} and X_{in} but they are shifted $\lambda/4$ from the end at $z=s$. Curves for G_{in} and B_{in} for the same lines and terminations as were used in computing R_{in} and X_{in} in Figs. 2–5 are shown in Figs. 8–11.

Dimensionless ratio factors corresponding to those defined in (19a) and (19b) are readily written down. Thus,

$$g_{1in}=R_c(G_{in}-\phi B_{in})=\frac{\sinh 2A_s}{\cosh 2A_s-\cos 2F_s'}, \tag{45}$$

$$b_{1in}=R_c(B_{in}+\phi G_{in})=\frac{\sin 2F_s'}{\cosh 2A_s-\cos 2F_s'}. \tag{46}$$

In most cases one may write

$$g_{1in}\doteq R_cG_{in}; \quad b_{1in}\doteq R_cB_{in}. \tag{47}$$

If (45) and (46) are solved for A_s and F_s' one has

$$A_s=\tfrac{1}{2}\tanh^{-1}\left[\frac{2g_{1in}}{g_{1in}^2+b_{1in}^2+1}\right], \tag{48}$$

$$F_s'=\tfrac{1}{2}\tan^{-1}\left[\frac{2b_{1in}}{g_{1in}^2+b_{1in}^2-1}\right]. \tag{49}$$

Values of F_s' for all possible values of g_{1in} and b_{1in} are shown schematically in Fig. 6. A comparison of these expressions with (21) and (22) shows that exactly the same circle diagram must apply to A_s and F_s' in terms of g_{1in} and $-b_{1in}$ as

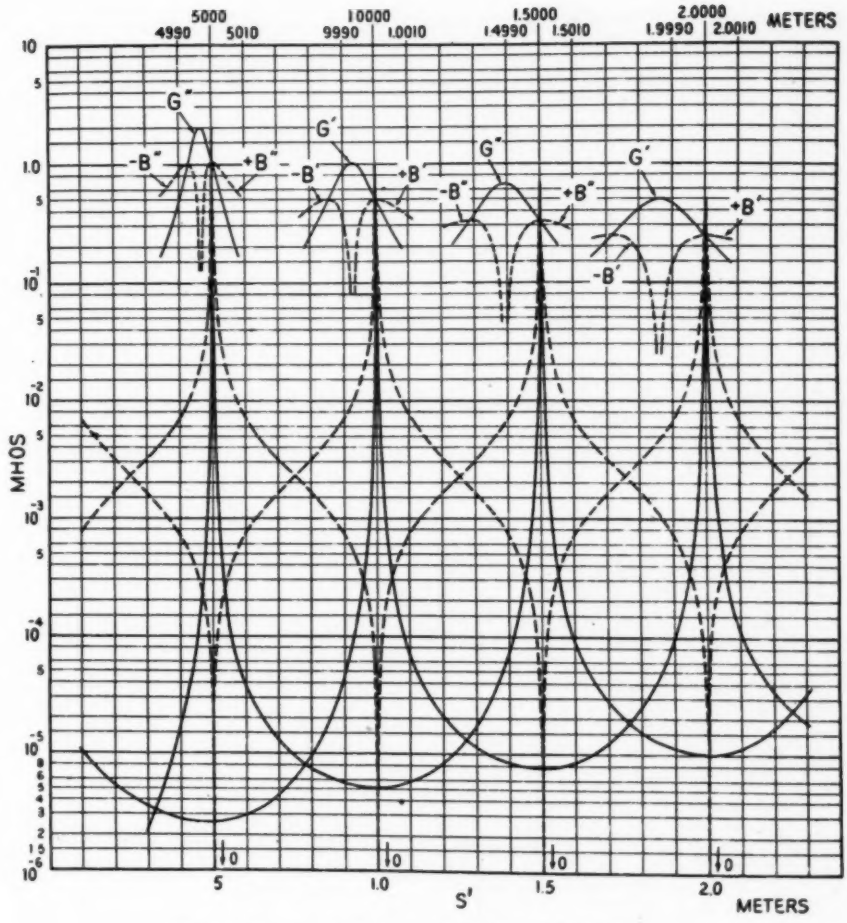

FIG. 8. Input admittance of the line of Fig. 2.
$Y_{in}' = 1/Z_{in}' = G_{in}' - jB_{in}'$ for bridged end (same wire).
$Y_{in}'' = 1/Z_{in}'' = G_{in}'' - jB_{in}''$ for open end.
The notation and constants of Fig. 2 apply.

to A_s and F_s in terms of r_{1in} and x_{1in}. This is indicated in Fig. 7.

For computing G_{in} and B_{in} from g_{1in} and b_{1in} the following formulas are useful:

$$G_{1in} = \frac{g_{1in} + \phi b_{1in}}{R_c(1+\phi^2)}, \tag{50}$$

$$B_{1in} = \frac{b_{1in} - \phi g_{1in}}{R_c(1+\phi^2)}. \tag{51}$$

By writing ρ_s for A_s, Φ_s' for F_s', and changing the subscript *in* to s, all of the above formulas apply to the terminal admittance, $Y_s = G_s - jB_s$, as well as to the input admittance of a terminated section of line.

Critical Values of the Input Admittance

Subject to the same conditions specified in (25) and (26), special formulas corresponding to (27)–(35) for the input impedance may be written for the input admittance. They are obtained directly from (27)–(35) by writing G for R, $-B$ for X, F' for F, and $-\phi$ for ϕ, and by dividing by R_c^2. Thus:

Simple formulas in a restricted range:

$$G_{in} = (1/R_c)[A_s \csc^2 F_s' + \phi \cot F_s'], \tag{52}$$

$$B_{in} = (1/R_c) \cot F_s', \tag{53}$$

provided

$$\sin^2 F_s' \gg A_s^2. \tag{54}$$

Conditions for input resonance and anti-resonance:

Input resonance:

$$B_{in}=0;\quad G_{in}\text{ at maximum} \tag{55a}$$

$$\text{at } \beta s=n\pi-\Phi_s'. \tag{55b}$$

Input anti-resonance:

$$B_{in}=0;\quad G_{in}\text{ at minimum}$$

$$\text{at } \beta s=(2n+1)\frac{\pi}{2}-\Phi_s'. \tag{55c}$$

Extreme values:

$$(G_{in})_{\min}=A_s/R_c \tag{56a}$$

$$\text{at } \beta s=(2n+1)\frac{\pi}{2}-\Phi_s'. \tag{56b}$$

$$(G_{in})_{\max}=1/R_cA_s \tag{57a}$$

$$\text{at } \beta s=n\pi-\Phi_s'. \tag{57b}$$

$$(B_{in})_{\min}=-\frac{1}{2R_cA_s}\left(1-\frac{\alpha}{\beta}\right) \tag{58a}$$

$$\text{at } \beta s=(n\pi-\Phi_s')\left(1+\frac{\alpha}{\beta}\right)+\rho_s. \tag{58b}$$

$$(B_{in})_{\max}=\frac{1}{2R_cA_s}\left(1+\frac{\alpha}{\beta}\right) \tag{59a}$$

$$\text{at } \beta s=(n\pi-\Phi_s')\left(1-\frac{\alpha}{\beta}\right)-\rho_s. \tag{59b}$$

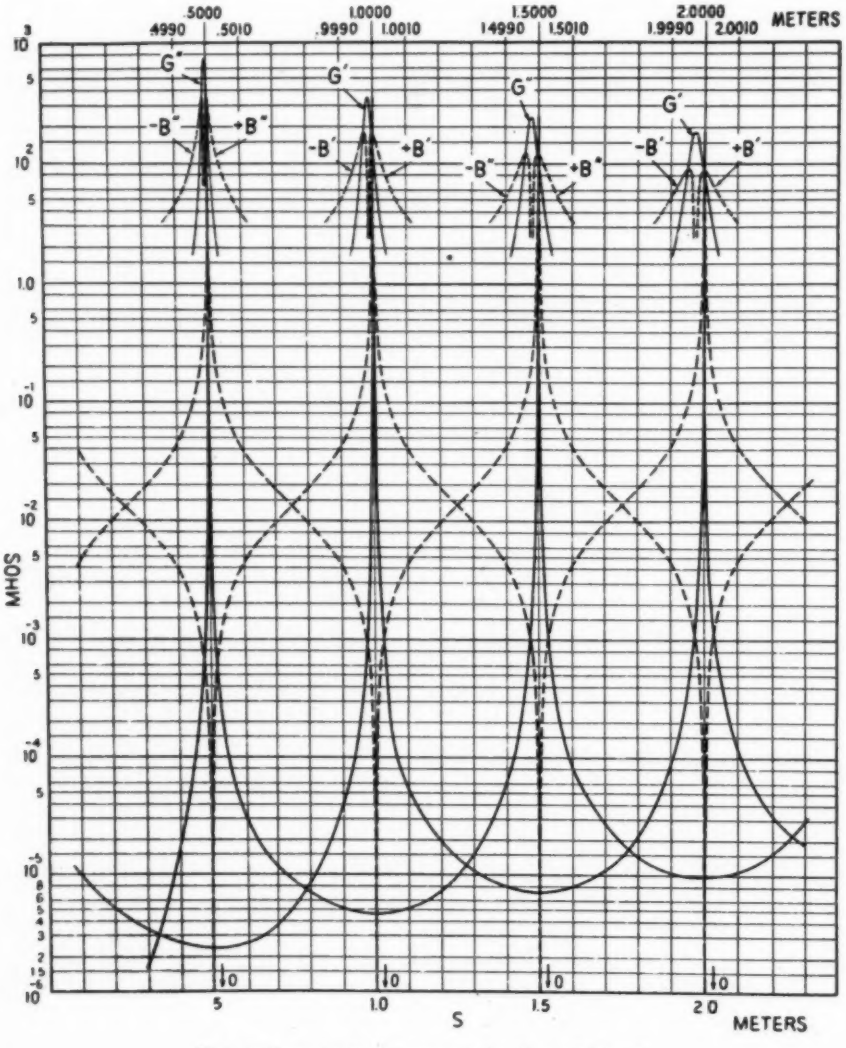

FIG. 9. Input admittance of the line of Fig. 3.

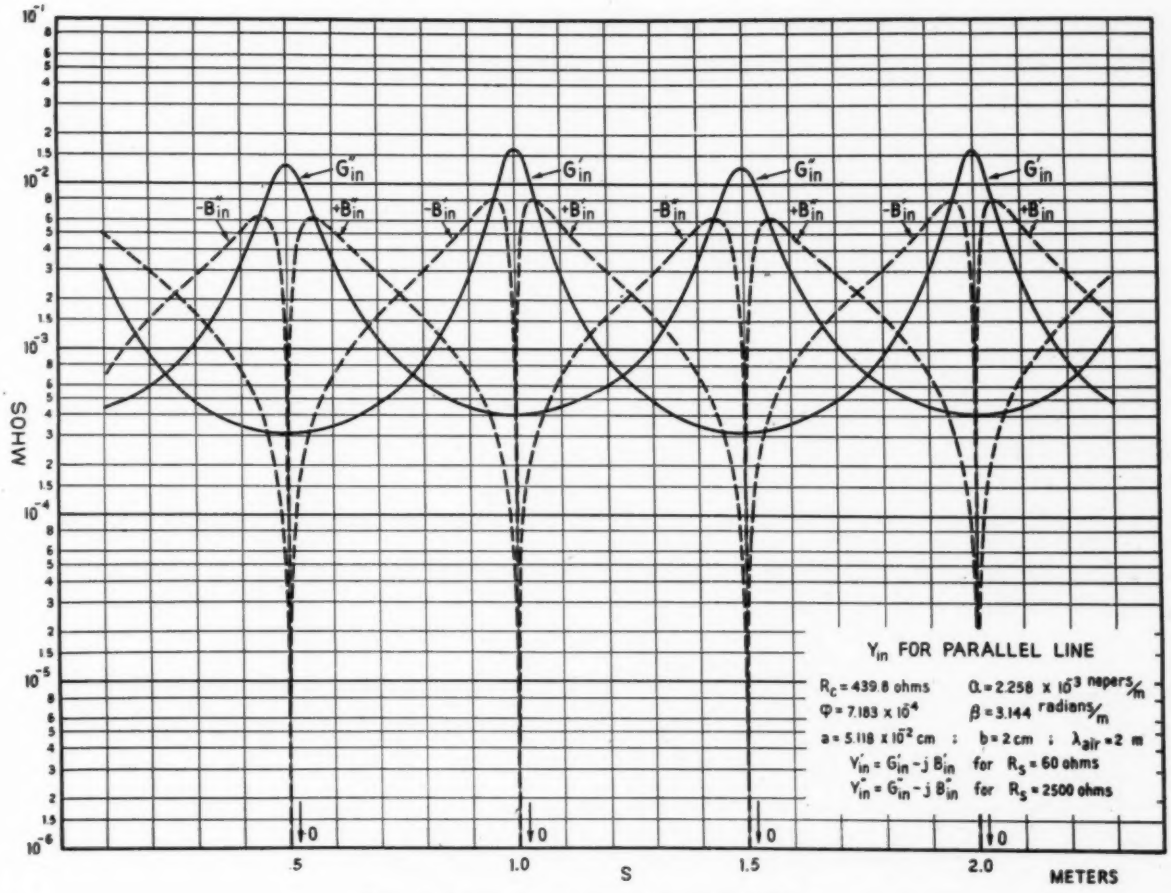

FIG. 10. Input admittance of the line of Fig. 4.

(α/β in the expressions (58a) and (59a) for extreme values of B_{in} is a term where, according to (26), α/β was substituted for ϕ, whereas α/β in formulas (58b) and (59b) is actually α/β and not ϕ. Hence a change in the sign of ϕ involves only α/β in (58a) and (59a).)

Special Values of the Terminal Functions

It has been pointed out that by a simple change of symbols (writing ρ_s for A_s, Φ_s for F_s, Φ_s' for F_s') and subscripts (s for in) all of the general formulas relating the input resistance and reactance, R_{in} and X_{in} (or conductance and susceptance, G_{in} and B_{in} with the over-all attenuation and phase functions, A_s and F_s or F_s', may be used to express the relations between the terminal resistance and reactance, R_s and X_s, (or conductance and susceptance, G_s and X_s, and the corresponding attenuation and phase functions, ρ_s and Φ_s or Φ_s'. Thus the schematic diagram of Fig. 7 indicates the quadrants of Φ_s or Φ_s' for given values of r_{1s} and x_{1s} or g_{1s} and b_{1s}. A number of important special forms of the terminal impedance have particularly simple terminal functions. These are outlined below without detailed analytical derivation. They follow directly from the general formulas by suitable specialization.

1. Predominantly Resistive Termination

The defining condition is

$$x_1=0 \quad \text{or} \quad X_s+\phi R_s=0. \tag{60}$$

This leads exactly to:

$$b_{1s}=0, \tag{61}$$

$$r_{1s}=1/g_{1s}=R_s/R_c. \tag{62}$$

The terminal functions are discontinuous at

$R_s = R_c$; with $n = 0, 1, 2, \cdots$, $\Phi_s' = \Phi_s \pm \pi/2$ one has:

$$\begin{cases} r_{1s} < 1: & \rho_s = \tanh^{-1} r_{1s}; \quad (63a) \\ \Phi_s = \pi/2 + n\pi; & \Phi_s' = n\pi. \quad (63b) \end{cases}$$

$$\begin{cases} r_{1s} = g_{1s} = 1: & \rho_s = \infty; \quad (63c) \\ \Phi_s = 3\pi/4 + n\pi; & \Phi_s' = \pi/4 + n\pi. \quad (63d) \end{cases}$$

$$\begin{cases} g_{1s} < 1: & \rho_s = \tanh^{-1} g_{1s}; \quad (63e) \\ \Phi_s' = \pi/2 + n\pi; & \Phi_s = n\pi. \quad (63f) \end{cases}$$

The function ρ_s is plotted as a function of r_{1s} and g_{1s} in Fig. 12. The conditions, $x_{1s} = 0$, $r_{1s} = 1$ are equivalent to $Z_s = Z_c$, the defining relation for a non-resonant line. It is clear from (14) and (15) that the input impedance of a line with $A_s = \alpha s + \rho_s = \infty$ is always Z_c. It is evidently immaterial whether A_s is made infinite by allowing s to become infinite (an academic case) or by allowing ρ_s to become infinite. It is seen from Fig. 12 that $R_s = R_c$ within 3 percent if ρ_s equals or exceeds 2. For all practical purposes a line for which ρ_s equals or exceeds 2 is non-resonant.

If one requires

$$X_s = 0 \quad (64a)$$

instead of $x_1 = 0$ as in (60) a more complicated situation arises unless $\phi = 0$. The formulas for ρ_s are unchanged, but Φ_s (or Φ_s') is not discontinuous. This is illustrated in Fig. 13. For series resonance ($X_s = 0$; $R_s \ll R_c$) one has approximately

$$\rho_s \doteq R_s/R_c; \quad \Phi_s \doteq \pi/2 + n\pi; \quad \Phi_s' \doteq n\pi. \quad (64b)$$

For parallel resonance (or a very high resistance leak of negligible reactance) ($X_s \doteq 0$, $R_s \gg R_c$) one

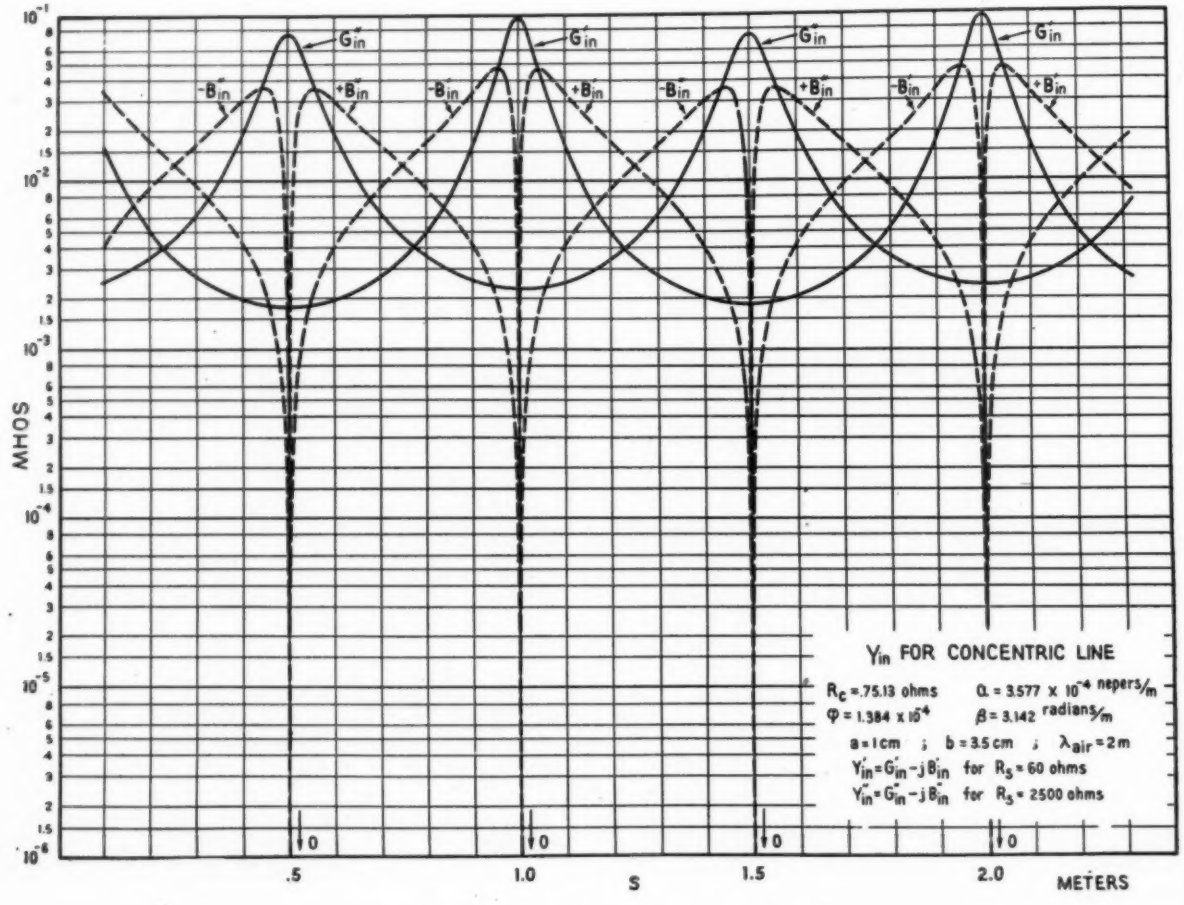

FIG. 11. Input admittance of the line of Fig. 5.

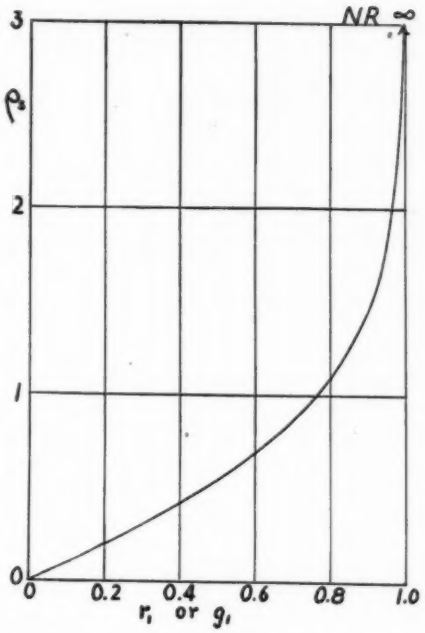

FIG. 12. Attenuation function ρ_s for a termination for which $x_{1s}=0$ or $X_{1s}=0$. Since $r_1=1/g_1$ in this case, the entire range of r_1 or g_1 from zero to infinity is included. The value for a non-resonant line is NR at $r_1=g_1=1$; $\rho_s=\infty$.

has approximately:

$$\rho_s \doteq R_c/R_s; \quad \Phi_s \doteq n\pi; \quad \Phi_s' = \pi/2. \tag{64c}$$

2. Predominantly Reactive Termination

Let it be required that *at least one* of the following two conditions be fulfilled:

$$r_{1s}^2 \ll |x_{1s}|^2, \tag{65a}$$

$$r_{1s}^2 \ll 1, \tag{65b}$$

or at least one of these:

$$g_{1s}^2 \ll |b_{1s}|^2, \tag{66a}$$

$$g_{1s}^2 \ll 1. \tag{66b}$$

If (65a) or (65b) is true, then

$$\rho_s = r_{1s}/(1+x_{1s}^2), \tag{67a}$$

$$\Phi_s = n\pi + \pi/2 + \tan^{-1} x_{1s}. \tag{67b}$$

If (66a) or (66b) is true, then:

$$\rho_s \doteq g_{1s}/(1+b_{1s}^2), \tag{68a}$$

$$\Phi_s' = n\pi + \pi/2 - \tan^{-1} b_{1s}. \tag{68b}$$

In both cases ρ_s is small. Curves for Φ_s and Φ_s' are shown in Fig. 14 as functions of x_1 and b_1.

Subject to (65) one can write

$$x_{1s} = X_s/R_c = \tan(\Phi_s - \pi/2) = \tan \Phi_s'. \tag{69}$$

This is the fundamental formula for the calculation of reactance from values of Φ_s or Φ_s' determined experimentally by methods described at a later point. It is valid only subject to (65).

The case of an open end on a conventional line with sufficiently close spacing to make radiation negligible ($\beta b \ll 1$) is contained in (66). Thus with $b_{1s}=0$ and $g_{1s}=0$ (68) gives:

$$\rho_s = 0; \quad \Phi_s' = n\pi + \pi/2; \quad \Phi_s = n\pi. \tag{70}$$

3. Highly Conducting Piston

It can be shown for coaxial (or hollow metal) pipes that a highly conducting piston which completely closes the pipe permits writing to a good approximation:

$$\rho_s \doteq 0; \quad \Phi_s \doteq \pi/2; \quad \Phi_s' \doteq 0. \tag{71}$$

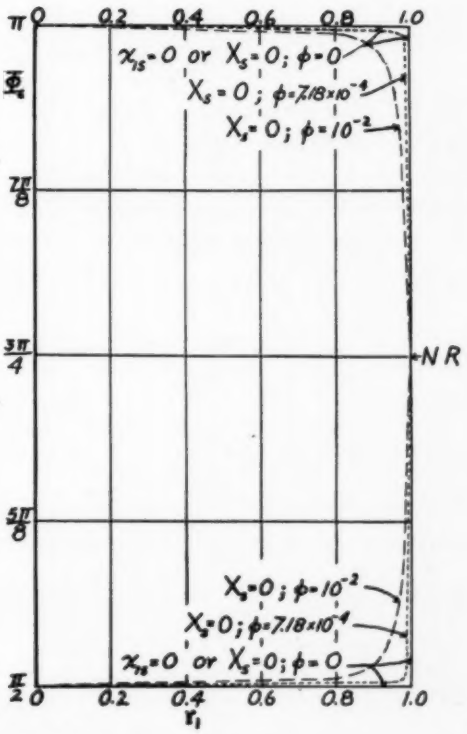

FIG. 13. Terminal function Φ_s for terminations for which $x_{1s}=0$ for any value of ϕ (solid line) or for which $X_{1s}=0$ with $\phi=0$ (solid line), $\phi=7.18\times10^{-4}$ (dotted line), $\phi=10^{-2}$ (dashed line). Above the line, $3\pi/4$, the g_1 scale, below this line the r_1 scale must be used. $r_1=1/g_1$ in the curves shown. The condition for a non-resonant line is $\Phi_s=3\pi/4$ with $r_1=g_1=1$ as indicated by the point NR. The g_1 and r_1 scales are the same.

4. *Wire Bridge*

A wire bridge made of the same wire as a parallel wire line and placed at right angles across the end of such a line has the following terminal functions. The wire spacing is b, its radius a.

$$\rho_s=\alpha(b-k_s);\quad \Phi_s=\pi/2+\beta k_s;\quad \Phi_s'=\beta k_s. \tag{72}$$

If L_s is the inductance of the wire bridge

$$k_s=\frac{L_s}{l}=\frac{b}{2}\left[\frac{\sinh^{-1}(b/a)+a/b-1+a^2/b^2}{\cosh^{-1}b/2a}\right]. \tag{73}$$

A plot of the function in brackets shows that to a very close approximation it remains constant at 0.95 for all values of b/a equal to or larger than 5. For most practical purposes it may be set equal to unity. In this case,

$$\rho_s\doteq\tfrac{1}{2}\alpha b;\quad \Phi_s=\pi/2+\tfrac{1}{2}\beta b;\quad \Phi_s'\doteq\tfrac{1}{2}\beta b. \tag{74}$$

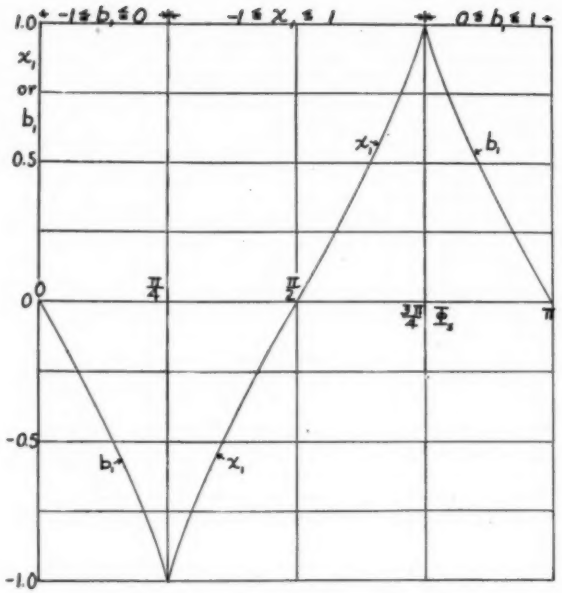

FIG. 14. The terminal function, Φ_s, for a termination satisfying one of (65a, 65b). Since $b_1=1/x_1$ the entire range is included.

The Distribution of Current and Potential Difference

The complex distribution of current and potential difference along a line which is driven at one end (V^e at $z=0$) is given in (8b) and (8c). If the line is driven by a loosely coupled oscillator which sets up a field which is cosine-symmetrical with respect to the points $z=x$ along the line so that it is equivalent to a pair of point generators at $z=x$,[2,3] it is merely necessary to replace θ_0 in (8b) and (8c) by (θ_0+kx) and write $s-x$ for s with s the new over-all length. One then has:

$$I_z=\frac{V_x^e}{Z_c}\left[\frac{\sinh(kx+\theta_0)\sinh(kw+\theta_s)}{\sinh(ks+\theta_0+\theta_s)}\right], \tag{75a}$$

$$V_z=\frac{V_x^e}{Z_c}\left[\frac{\sinh(kx+\theta_0)\cosh(kw+\theta_s)}{\sinh(ks+\theta_0+\theta_s}\right]. \tag{75b}$$

If the loosely coupled oscillator sets up a sine-symmetrical field so that it is equivalent to two pairs of point generators symmetrically placed with respect to x[2,3] one has:

$$I_z=\frac{W_x^e}{Z_c}\left[\frac{\cosh(kx+\theta_0)\sinh(kw+\theta_s)}{\sinh(ks+\theta_0+\theta_s)}\right], \tag{76a}$$

$$V_z=W_x^e\left[\frac{\cosh(kx+\theta_0)\cosh(kw+\theta_s)}{\sinh(ks+\theta_0+\theta_s)}\right]. \tag{76b}$$

Here W_x^e is an appropriate amplitude factor in volts. If the oscillator is unsymmetrical, a sum of (75) and (76) must be used.

[2] R. King, Proc. I.R.E. **28**, 84–87 (February 1940).

[3] R. King, Rev. Sci. Inst. **10**, 325–331 (1939).

In practice one is concerned particularly with the magnitude $|I_z|$ or $|V_z|$. These are obtained directly from (8b, c), (75a, b), or (76a, b) by noting that:

$$|\sinh(u+jv)| = [\tfrac{1}{2}(\cosh 2u - \cos 2v)]^{\frac{1}{2}} = [\sinh^2 u + \sin^2 v]^{\frac{1}{2}}, \tag{77a}$$

$$|\cosh(u+jv)| = [\tfrac{1}{2}(\cosh 2u + \cos 2v)]^{\frac{1}{2}} = [\sinh^2 u + \cos^2 v]^{\frac{1}{2}}. \tag{77b}$$

With these values the magnitudes $|I_z|$ and $|V_z|$ for the several cases given above are obtained directly in precisely the same form previously derived[4] by a much longer method. Important special cases are discussed in the earlier analysis. For convenient reference and as an illustration, the magnitude of (8b) is written below:

$$|I_z| = \left|\frac{V_0^e}{Z_c}\right| \left[\frac{[\sinh^2 \rho_0 + \sin^2 \Phi_0][\sinh^2(\alpha w + \rho_s) + \sin^2(\beta w + \Phi_s)]}{\sinh^2(\alpha s + \rho_0 + \rho_s) + \sin^2(\beta s + \Phi_0 + \Phi_s)}\right]^{\frac{1}{2}}. \tag{78}$$

The condition for resonance (maximum value of $|I_z|$ or $|V_z|$) is in all cases:

$$F_{0s} \equiv (\beta s_n + \Phi_0 + \Phi_s) = n\pi \tag{79a}$$

provided:

$$[(\alpha/\beta)\sinh(\alpha s + \rho_0 + \rho_s)]^2 \ll 1. \tag{79b}$$

The width Δs_n of a resonance curve between lengths for which $|I_z|^2$ has one-half the maximum value may be derived from (78). It is contained in:

$$\beta \Delta s_n/2 = \alpha s_n + \rho_0 + \rho_s = A_{0s} \tag{80a}$$

provided

$$(\tfrac{1}{2}\Delta s_n)^2 \ll s_n^2. \tag{80b}$$

The relations (79a) and (80a) are of fundamental importance in the experimental measurement of α, β, ρ, and Φ as well as of λ as will be shown below.

PART II: PRACTICAL APPLICATIONS

Space does not permit a detailed analysis of the several applications of transmission-line theory in the completely hyperbolic form that will be outlined below. The sequence of steps in each case follows directly from the general formulas.

The Transfer of Power and Efficiency of Transmission

A transmission line of length s has a load impedance Z_s. The power supplied at the input and of the line

$$P_0 = |I_0|^2 R_{in}. \tag{81}$$

The power supplied to the load is

$$P_s = |I_s|^2 R_s. \tag{82}$$

The fraction of power transferred to the load is:

$$\frac{P_s}{P_0} = \left|\frac{I_s}{I_0}\right|^2 \frac{R_s}{R_{in}}. \tag{83}$$

Using (78) expressed in double-angle form using (77a), and with $w = s - z$ (suitably specialized to $z = 0$ and $z = s$) and (14) (written both for R_{in} and for R_s) one obtains directly:

$$\frac{P_s}{P_0} = \left(\frac{\cosh 2A_s - \cos 2F_s}{\cosh 2\rho_s - \cos 2\Phi_s}\right)\left(\frac{\sinh 2\rho_s - \phi \sin 2\Phi_s}{\cosh 2\rho_s - \cos 2\Phi_s}\right) \times \left(\frac{\cosh 2A_s - \cos 2F_s}{\sinh 2A_s - \phi \sin 2F_s}\right)$$

or

$$\frac{P_s}{P_0} = \frac{\sinh 2\rho_s - \phi \sin 2\Phi_s}{\sinh 2(\alpha s + \rho_s) - \phi \sin 2(\beta s + \Phi_s)}. \tag{84}$$

This is the complete, general expression from which P_s/P_0 is easily computed. In all practical cases where a load of appreciable magnitude is

[4] R. King, Proc. I.R.E. 29, 640–648 (December, 1941). Formulas (84a, b, c, d), (49a, b, c, d).

involved, ρ_s is sufficiently large that the two terms in ϕ are entirely negligible. Accordingly the practical working formula is:

$$P_s/P_0=\sinh 2\rho_s/\sinh 2(\alpha s+\rho_s). \tag{85}$$

This is a formula of fundamental importance which permits a rapid and precise determination of line loss and efficiency of transmission. It applies to lines of any length and with any termination that satisfies:

$$\sinh 2\rho_s \gg \phi. \tag{86}$$

Since ϕ is of order of magnitude 10^{-4}, this is no restriction in practice.

The condition for *maximum* efficiency of the line is easily obtained from (85) by differentiating with respect to ρ_s. If this is done one obtains the familiar condition for a non-resonant line, *viz.*,

$$\rho_s=\infty. \tag{87a}$$

Then:

$$P_s/P_0=e^{-2\alpha s}. \tag{87b}$$

The db loss is:

$$\text{Loss (db)}=10\log_{10}(P_0/P_s). \tag{88}$$

The Standing Wave Ratio

The standing wave ratio, S, may be defined in terms of current ratios or voltage ratios as follows:

$$S_I=\frac{|I_z|_{\max}}{|I_z|_{\min}};\quad S_V=\frac{|V_z|_{\max}}{|V_z|_{\min}}. \tag{89}$$

In order to make S_I and S_V single-valued it will be understood that the distance $w=s-z$ will be kept as small as possible unless otherwise specified. Using (78) maximized and minimized with respect to w one obtains

$$S_I=\frac{\cosh(\alpha w_{\max}+\rho_s)}{\sinh(\alpha w_{\min}+\rho_s)}. \tag{90}$$

Here,

$$w_{\min}=n\lambda/2-\Phi_s/\beta, \tag{91a}$$

$$w_{\max}=\left(\frac{2n+1}{2}\right)\lambda-\frac{\Phi_s}{\beta}, \tag{91b}$$

with n chosen independently as small as possible in each case. (The formula for S_V is the same as (90), but the subscripts max and min in (91a, b) must be interchanged.) For lines with low attenuation which are used to drive an actual power-absorbing load $w_{\max}$ and $w_{\min}$ are always small enough—they need not exceed $\lambda/2$—so that one can write

$$\alpha s \ll \rho_s. \tag{92a}$$

Then:

$$S=\coth \rho_s. \tag{92b}$$

This is a formula of considerable importance and one of great simplicity. Upon referring to Table I one notes that the intercepts on the r_1 axis of the circle diagram of Fig. 7 are:

$$r_{1s}=1/S,\quad x_{1s}=0 \text{ and } r_{1s}=S;\quad x_{1s}=0. \tag{93}$$

Accordingly S is given directly by the value of r_{1s} where a circle of constant ρ_s crosses the r_{1s} axis for $r_{1s}>1$.

For a purely resistive termination one has with (63a, c, e)

$$S=g_{1s}=R_c/R_s \text{ for } g_{1s}>1, \tag{94a}$$

$$S=1 \text{ for } r_{1s}=g_{1s}=1, \tag{94b}$$

$$S=r_{1s}=R_s/R_c \text{ for } r_{1s}>1. \tag{94c}$$

The Spacing of Dielectric Supports Along Transmission Lines

Parallel wire lines and the inner conductor of coaxial lines are commonly supported by dielectric insulators in one form or another. If these are closely spaced compared with the wave-length they change the average capacitance and the leakage conductance per loop unit length of the line. If they are spaced at intervals comparable with a wave-length and the line is many wave-lengths long it is important to space the insulators in such a way that the input impedance of the loaded line is changed as little as possible by their presence. If the insulators are thin and made of a low loss dielectric they may be treated as equivalent to a small added capacitance across the line. In the case of a coaxial line with dielectric disks of thickness δ and of dielectric constant ϵ the *added* capacitance is:

$$C'=c\delta(\epsilon-1). \tag{95}$$

If the line is terminated with $Z_s=Z_c$ so that it would be non-resonant without the dielectric supports, the admittance of the line beyond the

last insulator, *viz.*, $Y_1=1/Z_c=(1+j\phi)/R_c(1+\phi^2)$, is in parallel with the susceptance $-\omega C'$. This parallel combination is the termination for the section of line at the next to the last insulator at a distance s_1 from the last one. Using (42) and (43) to determine the input susceptance and conductance, B_2 and G_2, of this section in parallel with $-\omega C'$, one can write:

$$B_2-\omega C'=-\phi/R_c(1+\phi^2), \tag{96a}$$

$$G_2=1/R_c. \tag{96b}$$

If s_1 is determined so that these equations are satisfied, the input impedance of the line beyond the next to the last insulator in parallel with this will equal Z_c.

The analysis shows that this will be true to an excellent approximation for a highly conducting line if

$$s_1=\lambda/4-(\delta/4)(\epsilon-1), \tag{97}$$

provided δ is a very small fraction of a wavelength. If the termination presented by the next to the last insulator and what lies beyond it is Z_c, then the problem of the next two insulators is a repetition of the one just solved. That is, for a non-resonant line the insulators should be arranged in pairs which are spaced slightly less than a quarter wave-length apart. The distances between successive pairs is immaterial.

A similar analysis shows that for a resonant line insulators should be placed at current maxima.

Impedance Transforming and Matching Sections

An important problem in line theory is the design of impedance transforming or matching sections which are inserted between a load impedance Z_s and a long line, so that this will be terminated in its characteristic impedance, $Z_c'=R_c'(1-j\phi')$. The required conditions are, then, that the input impedance of the impedance-transforming section when terminated in the load Z_s shall be precisely Z_c'. That is, one must have in most general terms:

$$R_{in}=R_c'; \quad X_{in}=-\phi' R_c' \tag{98}$$

or, in admittance form,

$$G_{in}=\frac{1}{R_c'(1+\phi'^2)}; \quad B_{in}=\frac{-\phi'}{R_c'(1+\phi'^2)}. \tag{99}$$

In all cases in which high efficiency of transmission is required, a matching section must be so designed that the power losses in it are small compared with the power transferred to the load. This means that the attenuation function ρ of the load must be very large compared with the product of α and the length of each section of transmission line used in the matching section. This is easily accomplished in practical cases if only short sections of low loss lines are used. It is then an excellent approximation to neglect all terms in the input impedances that contain ϕ' as a factor.

The *series transformer* is the simplest, although by no means the most flexible impedance transforming section. It consists merely of a line of length x and characteristic impedance Z_c as shown in Fig. 15. If the conditions stated above

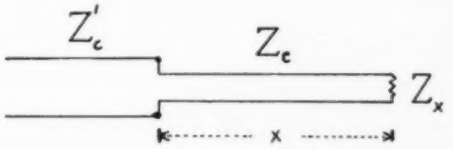

FIG. 15. Circuit of the series transformer for a two-wire line. Z_c' is the characteristic impedance of the long line.

are assumed satisfied, in particular, if one can write

$$\alpha x \ll \rho_x, \tag{100}$$

it follows directly from (14) and (15) substituted in (98) that one must have:

$$R_c'>R_c\begin{cases}\beta x=n\pi-\Phi_x, & \text{(101a)}\\ R_c'=R_c\coth\rho_x=R_cS, & \text{(101b)}\end{cases}$$

$$R_c'<R_c\begin{cases}\beta x=\dfrac{2n+1}{2}\pi-\Phi_x, & \text{(102a)}\\ R_c'=R_c\tanh\rho_x=R_c/S. & \text{(102b)}\end{cases}$$

Here n must be chosen as small as possible. For a purely resistive termination, i.e., with $Z_x=R_x$, it follows using (63a, b, e, f) and the smallest value of n that both of the above sets of conditions reduce to those of the familiar $\lambda/4$ trans-

former:

$$\begin{cases} \beta x = \pi/2, & (103a) \\ R_c' = R_c^2/R_x. & (103b) \end{cases}$$

(The one-to-one transformer is also contained in (101) and (102).) The series transformer is limited as an impedance transforming device by the practically available magnitudes of the characteristic impedance of a transmission line.

The *single movable stub* is a simple and extremely flexible impedance transforming device. The circuit arrangement is shown in Fig. 16 and

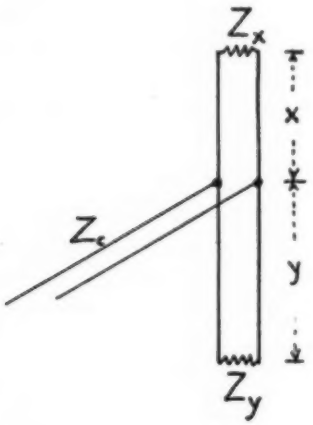

FIG. 16. Circuit of a single movable stub. Z_x is the load. Z_c is the characteristic impedance of the long line.

the condition for match is

$$Y_{in} = Y_{inx} + Y_{iny} = Y_c, \quad (104)$$

where Y_{in} is the admittance terminating the long line with characteristic admittance Y_c. Upon separating real and imaginary parts and introducing the ratio factors defined in (45) and (46) one has:

$$(g_{1in})_x + (g_{1in})_y = 1, \quad (105a)$$

$$(b_{1in})_x + (b_{1in})_y = 0. \quad (105b)$$

If the terminating impedance, Z_y, (which is entirely at one's disposal), is designed with ρ_y equal to zero or extremely small, and if the length y is kept below a half-wave-length one can readily satisfy the inequality:

$$(\alpha y + \rho_y) \ll 1. \quad (106)$$

Except for $(\beta y + \Phi_y)$ very near $\pi/2$ this permits writing

$$(g_{1in})_y \ll 1. \quad (107)$$

It also permits the use of the simple approximate formula (53) for $(B_{in})_y$ with the ratio factors defined by (47). In this case (105) reduces to:

$$(g_{1in})_x = 1, \quad (108a)$$

$$(b_{1in})_x = -\cot(\beta y + \Phi_y'). \quad (108b)$$

By using (45) and (46) subject to the condition

$$\alpha x \ll \rho_x, \quad (109)$$

and with

$$F_s' = \beta x + \Phi_x'; \quad A_s = \rho_x, \quad (110)$$

the following equations are obtained:

$$\cos 2(\beta x + \Phi_x') = \cosh 2\rho_x - \sinh 2\rho_x = (\cosh \rho_x - \sinh \rho_x)^2, \quad (111)$$

$$\cot(\beta y + \Phi_y') = -\left(\frac{\sin 2(\beta x + \Phi_x')}{\sinh 2\rho_x}\right). \quad (112)$$

These may be rearranged in several different ways including especially the following useful formulas.

$$\cot(\beta x + \Phi_x') = \pm(\coth \rho_x)^{\frac{1}{2}} = \pm S^{\frac{1}{2}}, \quad (113)$$

$$\cot(\beta y + \Phi_y') = \mp[(\coth \rho_x)^{\frac{1}{2}} - (\tanh \rho_x)^{\frac{1}{2}}] = \mp\left[\frac{S-1}{S^{\frac{1}{2}}}\right], \quad (114)$$

$$\cot(\beta x + \beta y + \Phi_x' + \Phi_y') = \mp(\coth \rho_x)^{\frac{1}{2}} = \mp S^{\frac{1}{2}}. \quad (115)$$

Here the upper signs go together, as do the lower ones. If Φ_x' and ρ_x or Φ_x' and S' are known from computation or experimental measurement, the lengths x and y or x and $x+y$ for the stub and its location are readily computed. They may also be determined from the circle diagram by a quasi-cut-and-try method to obtain a suitable combination to satisfy (105).

Insulating Stubs

At high frequencies it is often advantageous to use high impedance sections of two-wire or coaxial line as insulating supports. Such sections are easily designed so that the conditions (25) and (26) are fulfilled and also so that leakage conductance is negligible. The attenuation constant is then defined by the simple formula,

$$\alpha \doteq r/2R_c, \quad (116)$$

where r is the resistance per loop unit length of the line. The maximum input impedance of a section of length s is given in (33a).

$$(R_{in})_{max} = R_c/(\alpha s + \rho_s). \tag{117}$$

This value occurs according to (33b) when

$$\beta s = n\pi - \Phi_s. \tag{118}$$

The highest resistance for a given line with fixed values of b (wire spacing or inner radius of outer coaxial conductor) and a (radius of wire or of the inner coaxial conductor) occurs when the over-all attenuation of the section is a minimum. That is, when s and ρ_s in (117) are reduced as much as possible. This will be true if the end is bridged by a copper wire in the case of an open wire line or closed by a metal piston in a coaxial line. From (74) and (71) one has:

Wire bridge: $\rho_s \doteq \frac{1}{2}\alpha b$; $\Phi_s \doteq (\pi/2) + \frac{1}{2}\beta b$. (119)

Piston: $\rho_s \doteq 0$; $\Phi_s \doteq \pi/2$. (120)

With these values and $n=1$, (117) and (118) reduce to:

$$(R_{in})_{max} = R_c/\alpha s', \tag{121}$$

where

$$s' = \pi/2\beta = \lambda/4 \tag{122}$$

and λ is the wave-length on the line. For a two-wire line with wire bridge,

$$s' \doteq s + \tfrac{1}{2}b. \tag{123}$$

For a coaxial line

$$s' = s. \tag{124}$$

Using (122) and (116) in (121), one obtains:

$$(R_{in})_{max} = 8R_c^2/r\lambda. \tag{125}$$

For a two-wire line

$$R_c = (\mu_d/\epsilon_d)^{\frac{1}{2}}\, 120 \cosh^{-1}(b/2a); \quad r = m/a. \tag{126}$$

For a coaxial line

$$R_c = (\mu_d/\epsilon_d)^{\frac{1}{2}}\, 60 \ln(b/a); \quad r = (m/2b)(1+b/a). \tag{127}$$

In both cases,

$$m \equiv 2\left(\frac{f\mu_c \times 10^{-7}}{\sigma_c}\right)^{\frac{1}{2}} = 2\left(\frac{30\mu_c}{\lambda\sigma_c(\mu_d\epsilon_d)^{\frac{1}{2}}}\right)^{\frac{1}{2}} \tag{128}$$

with f the frequency, μ_c the relative permeability, σ_c the conductivity in reciprocal ohm-meters of the conductors; μ_d and ϵ_d the relative permeability and the dielectric constant of the dielectric medium between the conductors. If (127) or (128) is substituted in (125) one obtains

Two-wire line:

$$(R_{in})_{max} = \frac{bK}{\lambda^{\frac{1}{2}}}\left[\frac{2a}{b}\left(\cosh^{-1}\frac{b}{2a}\right)^2\right]. \tag{129}$$

Coaxial line:

$$(R_{in})_{max} = \frac{bK}{\lambda^{\frac{1}{2}}}\left[\left(\ln\frac{b}{a}\right)^2 \Big/ \left(1+\frac{b}{a}\right)\right], \tag{130}$$

with

$$K = \frac{\mu_d}{\epsilon_d}\frac{(240)^2}{m(\lambda)^{\frac{1}{2}}}. \tag{131a}$$

For copper conductors ($\sigma = 5.65\times10^7$ 1/ohm-m) with air as dielectric ($\epsilon_d = 1$, $\mu_d = 1$), so that

$$K = 3.95\times10^7. \tag{131b}$$

The functions in square brackets in (129) and (130) are plotted as functions of R_c in Fig. 17. The maxima of the functions are:

Two-wire line:

$$\frac{2a}{b}\left(\cosh^{-1}\frac{b}{2a}\right) = 1.062$$

$$\text{at } R_c = 246(\mu_d/\epsilon_d)^{\frac{1}{2}}; \quad b/a = 7.9. \tag{132}$$

Coaxial line:

$$\left(\ln\frac{b}{a}\right)^2 \Big/ \left(1+\frac{b}{a}\right) = 0.481$$

$$\text{at } R_c = 133.1(\mu_d/\epsilon_d)^{\frac{1}{2}}; \quad b/a = 9.2. \tag{133}$$

Accordingly the extreme values of the input resistance are:

Two-wire line:

$$(R_{in})_{maxmax} = \frac{1.062bK}{\lambda^{\frac{1}{2}}}; \quad a = 0.127b. \tag{134}$$

Coaxial line:

$$(R_{in})_{maxmax} = \frac{0.481bK}{\lambda^{\frac{1}{2}}}; \quad a = 0.109b. \tag{135}$$

By changing the wire radius from 0.05118 cm to $a = b/7.9 = 0.253$ cm the maximum input resistance of the parallel wire line represented in

Fig. 2 can be increased from the actual 397,000 ohms to the maximum possible with a 2-cm spacing, 595,000 ohms.

The Experimental Determination of the Parameters of the Line and its Terminations

The Measurement of the Phase Constant and Wave-Length

An accurate determination of β for a transmission line of any type (and hence of $\lambda=2\pi/\beta$) depends upon the condition for resonance (79a). If one of the two terminations of the line is movable it is possible to measure s_n for two or more resonance curves provided (79b) is satisfied. A really accurate determination is possible only if the widths of the resonance peaks are small so that (80b) is true. This requires that the two terminations be characterized by reasonably small values of ρ_0 and ρ_s, and that the length of the line be as small as possible. If this is true (79a) gives

$$\beta(s_2-s_1)=\pi, \tag{136a}$$

or

$$(s_2-s_1)=\lambda/2. \tag{136b}$$

From these relations β and λ may be determined as accurately as (s_2-s_1) can be measured. Very great precision is possible when the resonance peaks are sharp if plots are made of two successive peaks to a large scale and midpoint lines are drawn to locate the maxima. Really sharp peaks are possible only if the driving generator and a highly sensitive detector are both very loosely coupled.

The Measurement of the Attenuation Constant

Several experimental methods for determining α are theoretically available. These depend upon techniques which permit the use of formula (78), (80a), or (90). In practice, the method which makes use of (80a) is to be preferred because it involves the least difficult experimental procedure and the simplest formula. Thus, by measuring the widths, Δs_1 and Δs_m of two resonance peaks for lengths of line given by s_1 and s_m, (80a) leads to:

$$\alpha=\frac{\beta}{2}\left(\frac{\Delta s_m-\Delta s_1}{s_m-s_1}\right). \tag{137}$$

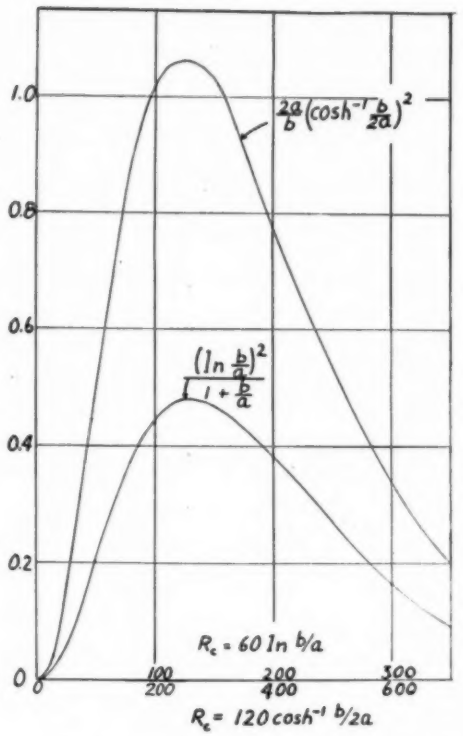

FIG. 17. The function $(2a/b)(\cosh^{-1} b/2a)^2$ as a function of $R_c=120\cosh^{-1}(b/2a)$ and the function $(\ln b/a)^2/(1+b/a)$ as a function of $R_c=60\ln(b/a)$.

All that is required experimentally is a careful plotting of two resonance peaks using a detector with an accurately calibrated scale. Attenuation due to the detector and the terminations may be included in ρ_0 or ρ_s, so that it is subtracted out provided no change is made in the location of the detector or the degree of coupling during the measurements. It is important to note that (137) presupposes that (79b) and (80b) are satisfied. This means that the attenuation of the line should be measured with a minimum load. If a highly sensitive detector and low loss terminations are used this is easily accomplished.

The Measurement of the Phase Function of a Termination

The function Φ_s for a termination may be determined very readily using (79a). Thus, let the resonant length s_1 be determined for the line with a termination at $z=s$ which has a known value of $\Phi_s(\Phi_s=\pi/2$ for a metal piston in a coaxial or hollow pipe; $\Phi_s=\pi/2+\beta k_s$ with k_s defined by (73) for a two-wire line). Then let the resonant

length $s_{1\Phi}$ be measured using a termination of unknown Φ_s. By subtracting the two equations (79a) obtained in this way, one has:

$$\Phi_s=\pi/2+\beta(s_1-s_{1\Phi}). \tag{138}$$

Φ_0 may be determined by the same method.

Since (79a) presupposes (79b), the relation (138) is useful only if (79b) is satisfied. If this is not the case it follows necessarily that $(\alpha s+\rho_s)$ is so large (2 or greater) that for practical purposes the line is equivalent to one terminated in its characteristic impedance. A knowledge of Φ_s is, then, not required.

The Measurement of the Attenuation Function for a Termination

If α for the line is known, the terminal function ρ_s may be determined by measuring the width Δs_1 of the first resonance curve with a termination of known ρ_s ($\rho_s=0$ for a metal piston; $\rho_s=\alpha(b-k_s)$ for a two-wire line with spacing b) and the widths $\Delta s_{1\rho}$ of the first resonance curve with the termination of unknown ρ_s. If the resonant lengths in the two cases are, respectively, s_1 and $s_{1\rho}$, (80a) may be written for the two cases and ρ_s determined by subtraction. The formula for the two-wire line is:

$$\rho_s=\tfrac{1}{2}\beta(\Delta s_{1\rho}-\Delta s_1)+\alpha(s_1-s_{1\rho})+\alpha(b-k_s). \tag{139}$$

For the metal piston the last term is omitted. It is important to note that the attenuation due to the detector is subtracted out, since it may be included in ρ_s. The location and the degree of coupling of the detector relative to the end at $z=0$ must be unchanged during the measurement.

The determination of ρ_s from (139) presupposes that both (79b) and (80b) are satisfied. Although (79b) is satisfied for values of ρ_s as large as 1 or 2, this is not true of (80b) which implies values of ρ_s which are relatively small. If (80b) is not satisfied because ρ_s is too large, the condition (92a) is easily fulfilled. It is then possible to determine ρ_s directly from the experimentally determined standing wave ratio using (92b) or, if necessary, (90). The measurement of the standing-wave ratio depends upon the use of a movable detector the deflection of which is read when it is a maximum and when it is a minimum. The effect of the coupled detector on the standing wave distribution along the line is entirely different at these two locations. It represents a larger load on the line where the deflection is maximum; the detuning effect is always greatest when the detector is coupled near a point of maximum concentration of charge. For "current detectors" these points do not occur together; for "voltage detectors" they do.

Since it is practically out of the question either to design a detector which neither loads nor detunes the line, or to obtain a simple correction for the effect of the detector in the two locations (this is discussed in reference 4), the value of ρ_s determined from the standing wave ratio is moderately accurate only for small standing-wave ratios, i.e., for large value of ρ_s. Accordingly (139) should be used as described above so long as (80b) is satisfied. Only when the width of the resonance curve exceeds the limitation imposed by (80b) should ρ_s be determined from (92b) or (90). The results are, then, necessarily approximate because no really accurate correction for the disturbing effect of the detector can be made easily.

"Q" for a Transmission Line

The quality factor "Q" for a resonant transmission line of any type may be defined by

$$Q=f_n/2\delta f=s_n/\Delta s_n \tag{140}$$

provided (80b) is satisfied or the following condition

$$(\delta f)^2\ll f_n^2. \tag{141}$$

Here δf is the change in frequency required to reduce the resonant amplitude of current or potential difference to 0.707 of its maximum at the resonant frequency f_n. Q cannot be defined satisfactorily unless (80b) and (141) are satisfied.

Upon substituting from (80a) into (140) one has:

$$Q=\beta s_n/2(\alpha s_n+\rho_0+\rho_s). \tag{142}$$

If the terminations are characterized by negligible values of ρ so that

$$\alpha s_n\gg(\rho_0+\rho_s) \tag{143}$$

one has

$$Q=\beta/2\alpha. \tag{144}$$

PART III. A GENERAL DEFINITION OF IMPEDANCE

It is beyond the scope of the present discussion to consider in detail the analysis of transmission circuits (or wave guides) constructed of hollow metal pipes or of coaxial pipes of large cross section. It has already been stated that such an analysis may be formulated in terms of appropriately defined stream functions (in amperes) and potential functions (in volts) from which the distribution of surface currents and charge may be determined by differentiation. These functions depend upon three coordinates. It is, however, always possible to separate the variables so that the dependence on the axial variable z is expressed by an equation like (1) with V a general potential function and I a general stream function defined throughout the dielectric in the interior of the transmission circuit. The parameters r, l, g, and c have quite different meanings in the general case, but they may be combined into a propagation constant, $k_g=\alpha_g+j\beta_g$, and a characteristic impedance, $Z_{cg}=R_{cg}(1-j\phi_g)$. (The subscript g is used to denote "guide.") The solution of the equations for the potential function and the stream function may be expressed in any one of the several ways outlined above by defining by symbolic analogy terminating impedances, coefficients of reflection, and terminal functions.

In spite of the formal equivalence between potential and stream functions in the general case and potential difference and total current in the special case of conventional lines, the total current is not a special case of the stream function. Thus if the general analysis is specialized to apply to the conventional coaxial line, *the stream function does not reduce to the total current.* It is still defined in the dielectric between the coaxial conductors, and the surface density of current must be obtained from it by differentiation. It follows directly that the ratio of the general potential function to the stream function for a circuit of infinite length, a quantity which correctly defines Z_{cg} (just as V/I for an infinite line gives Z_c for the line) is not at all equivalent to Z_c for a conventional line. In fact, Z_{cg} for all conventional coaxial lines regardless of the radii of the conductors, reduces to the pure resistance, 376.7 ohms. Accordingly the identity of mathematical form in the analysis of general and conventional transmission circuits does not imply a complete physical analogy in all of the analytically comparable parameters. Such an analogy does not exist in the case of the characteristic impedance; on the other hand, it does exist in the case of the propagation constant k_g which plays an exactly comparable role as k, and reduces to this when the general analysis is specialized to apply to the conventional line.

The formal equivalence between the general analysis and conventional line theory is limited to the dependence of the potential and stream functions on the axial variable. It follows that variations in the axial direction are analogous. For any given transmission circuit excited in a particular mode one can determine experimentally phase constant, β_g; wave-length, λ_g; attenuation constant, α_g; phase and attenuation functions of terminations, Φ_g and ρ_g; in entirely comparable ways for use in the same equations and formulas. The circuit will be characterized by traveling waves and a standing wave ratio of unity whenever it is terminated in a device with $\rho_{sg}=\infty$.

With ρ_g and Φ_g determined it is a simple step to define terminal and input impedance by using (9) and (10) and the value of Z_{cg} appropriate for the particular pipe. But such an impedance is not a generalization of conventional impedance; it does not reduce to it when specialized to ordinary transmission lines because Z_{cg} does not reduce to Z_c. So long as one is concerned with a particular pipe excited in one mode this is immaterial because all impedances involve the same factor Z_{cg}. However, when two pipes of different cross-sectional shape or two pipes of the same or different shape excited in a different mode are to be joined, meaningful results are not obtained in terms of the values of Z_{cg} for the two cases. In particular, if a transmission circuit which is terminated with $\rho_{ge}=\infty$ is connected as a load to a driven pipe with the same value of Z_{cg} but of different shape using a different mode, the driven pipe will not in general be terminated in $\rho_{sg}=\infty$.

The writer is indebted to Mr. Jack Porter for computing the data for Figs. 2–5 and Figs. 8–11, and to Mr. Charles W. Harrison, Jr. for computing the data for Fig. 7. Dr. A. H. Wing read the manuscript and made valuable suggestions.

Luminescence of Valve Metals During Electrolytic Oxidation

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I. INTRODUCTION

THE weak luminescence surrounding the aluminum anode in electrolytic rectifiers was first reported by Guntherschulze[1] in 1906. Since it is visible only after a certain threshold potential is applied, he regarded it as the result of a discharge through a gaseous layer trapped on the surface of the oxide coating. Many experimenters working with rectifiers have accepted this explanation as correct and hence have not undertaken studies which would reveal the mechanism involved. In so doing, they have failed to appreciate the implications of two important facts: (a) that the luminescent spectrum depends upon impurities within the metal, and (b) that when the electrolyte is such as to form an oxide coating upon the metal, the emitted spectrum is independent of the electrolyte.

The effect is not confined to aluminum. In fact, it is observed when magnesium, zinc, tungsten, or tantalum[2] is employed as the anode in various electrolytes—citric acid, oxalic acid, ammonium oxalate, phosphoric acid, dilute sulfuric acid, and many others. In connection with the luminescence, the significant characteristic of the acid is that it permits the formation of an oxide coating upon the anode.

When the oxide coating dissolves in the solution, the light is not observed. Because of this property, the emission is detected only with difficulty when either zinc or magnesium is the anode. It appears to be strongest when aluminum is used in oxalic acid. Therefore, this combination has been utilized in most investigations.

Even less publicized than the anodic luminescence of the above-mentioned metals is the fact that if the polarity is reversed when the anodic luminescence is visible, a momentary flash of light is emitted.[2] By visual observation this "cathodic flash" appears to be shifted further toward the red end of the spectrum than is the anodic glow. No further emission is seen although a large current continues to flow at a relatively low voltage. Furthermore, once the light has disappeared, it will not appear again, except in certain special cases, until after the electrode has been made anodic once more. If the polarity is again reversed, returning the positive polarity to the metal, the initial intensity of the light is rather high but decays quickly to the nearly constant value of the anodic glow. As a result, the luminescence is much stronger when being excited by alternating voltage than by direct current.

The occurrence of the cathodic flash militates against the supposition that the luminescence is a form of chemiluminescence. If it were, the reaction proceeding at the anode during the passage of current would have to be carried on for a short time with a reversed polarity. Within the electrolyte an oxidation takes place at the anode, but at the cathode only a reduction is possible. Therefore, it is exceedingly difficult to imagine a reaction at the aluminum anode that will continue even for a moment after it is made cathodic.

Moreover, the fact that the spectrum depends upon the metal and the impurities it contains indicates that the radiation is characteristic of the metal or its oxide. The spectra, however, have been of no assistance in determining the origin of the glow. Consequently, the author has made studies of the lifetime of both the anodic and cathodic flashes with the hope that they might reveal the processes by which the light is emitted. The present article proposes to show that these experiments, together with others previously reported, suggest that the origin of the luminescence is similar to that of the luminescence in the semi-conductor sulfide phosphors and is a property of the oxide coating formed on the metal. A qualitative picture based upon the

[1] A. Guntherschulze, Ann. d. Physik **21**, 929 (1906).
[2] J. S. Forrest, Phil. Mag. **10**, 1003 (1930); J. Lavaux, Comptes rendus **169**, 180 (1919); T. Rummel, Zeits. f. Physik **101**, 276 (1936).

modern theory of conduction in semi-conductors is proposed to explain the reported luminescent effects.

II. THE SPECTRUM

Spectroscopic investigations have been hindered by the low intensity of the radiation. As a result, workers have usually confined themselves to visual observations. Guntherschulze and Betz[3] and Betz[4] have attempted to correlate photometrically the intensity of the luminescence with current and voltage. Such relations are difficult to establish, because, as will be seen later, there appears to be no simple correspondence between luminescent intensity and the current flowing or the voltage impressed across the cell.

Recently Gumminski[5] completed an extensive spectroscopic investigation in which he studied the effect of various impurities and applied voltages. He concerned himself with only the anodic glow. He found that all spectra were continuous extending from λ3705 to λ5874 with a maximum near λ4590. The spectral energy density increased in the red when the voltage was raised, and manganese in the metal shifted the maximum toward the red. Other impurities exerted less effect.

In order to determine whether or not the cathodic flash originated from a source similar to that of the anodic flash, the spectra of both radiations were investigated by the author. In view of the facts that (a) the cathodic light is obtained only after the metal has been anodic, and (b) the anodic luminescence is brightest immediately after the metal has been cathodic, arrangements were made to obtain both spectra simultaneously. This was accomplished by synchronizing a commutator and shutter so that the upper half of the spectrograph slit was exposed when the metal was anodic and the lower half exposed when the metal was cathodic. The polarities were then alternated at a constant frequency.

Spectra 5a and 5b in Fig. 1 are for the aluminum alloy known as 3S (commercial aluminum plus 1.2 percent manganese) in an aqueous solution of 0.5 percent oxalic acid at 18°C. It is evident that both spectra are continuous and

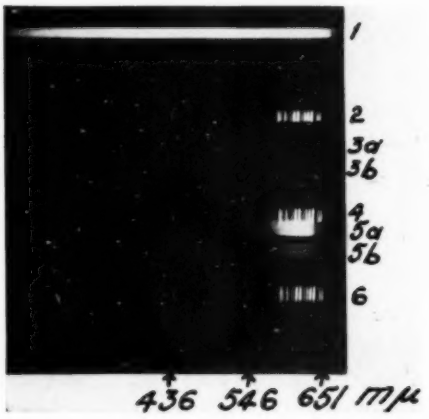

FIG. 1. Spectra of anodic and cathodic luminescence of aluminum. (1) Tungsten lamp. (2), (4), and (6) Mercury-neon discharge. (3a) and (3b) Cathodic and anodic luminescence of pure aluminum, respectively. (5a) and (5b) Cathodic and anodic luminescence of aluminum with 1.2 percent added manganese, respectively.

have approximately the same spectral range with the intensity increasing toward the longer wavelengths, but the cathodic flash appears to contain a larger percent of its spectral intensity in the red end than does the anodic flash, and it is also the brighter of the two. The film was not calibrated, hence no quantitative statements can be made. It may well be that all the difference in appearance is due to the greater intensity of the cathodic flash. The same behavior is found for very pure aluminum (99.976 percent aluminum, 0.007 percent iron, 0.01 percent copper, 0.007 percent silicon). Here the total intensity is so low, however, that even after five hours exposure, the anodic half is too weak to show in this print. Spectra of the type observed here are encountered quite frequently in the fluorescence of crystals.[6]

III. RATES OF DECAY OF LUMINESCENT FLASHES

The spectra assist very little in explaining the phenomena. They indicate only that manganese affects the emission spectra here as it does when included in solid phosphors, causing the lumi-

[3] A. Guntherschulze and H. Betz, Zeits. f. Physik **74**, 681 (1932).
[4] H. Betz, Zeits. f. Physik **95**, 189 (1935).
[5] K. Gumminski, Bull. Acad. Polon. Sci. Let. 3–4, p. 145 (1936).
[6] J. T. Randall, Trans. Faraday Soc. **35**, 2 (1939).

nescence to be yellow. On the other hand, decay curves of the fluorescence in such phosphors often have proved helpful in analyzing their behavior.[7] If the emission occurs during the transition of an atom to a ground state from an excited state, the rate of decay of the radiant intensity is given by the expression

$$I = I_0 e^{-\alpha t}. \tag{1}$$

When two separate particles such as an ion and electron unite to give off light, the rate of decay follows a more complex curve. For the special case in which both types of the reacting units are present in equal concentrations, we obtain the expression

$$I_0/I = (\alpha t + 1)^2. \tag{2}$$

In both expressions α is a quantity that depends upon the probability of radiation. Such considerations make it appear worth while to measure the rates of decay of not only the anodic glow following the removal of the potential, but also that of the cathodic and initial anodic flashes.

A phosphoroscope constructed to measure the decay rates photographically is illustrated in Fig. 2. The electrodes are immersed in the water-cooled acid solution contained in the glass vessel (G) equipped with a collimated slit (S) in the bottom. The aluminum electrode (Al) is shaped so that one side is about $\frac{1}{8}$ inch directly above the slit. The other electrode (B) employed in all the experiments to be described is carbon. The slit is placed just above the periphery of a fifteen-inch disk (D) on which are fastened two strips of film arranged to be exposed to the aluminum when the potentials are applied. The disk and a commutator (C) were turned simultaneously by a variable speed motor (M). The commutator consists of two sets of parallel copper strips fastened to the circumference of a wooden cylinder. One set is parallel to the axis of rotation, the other set is perpendicular to it. Contact with the copper is made by four graphite brushes (W) to which the electrical connections are made so that when the brushes are pressing against one set of strips the aluminum is anodic and when against the other it is cathodic. The circular sections of film are always cut from a large sheet, a step plate is added to a strip of it for calibration. The exposure of any particular spot on the film depends upon the relative width of the slit and the disk circumference. Therefore, while employing a 0.5-mm slit width, the speed of rotation is adjusted so that the exposed streak on the film is spread out far enough for ease in analyzing. The customary speed is one revolution per second and the apparatus is operated a sufficient length of time to give proper blackening of the film, generally about one hour for pure aluminum and from 10 to 15 minutes for commercial aluminum with 1.2 percent added manganese. The films are analyzed with the aid of a recording microphotometer. Thus we are able to determine the decay rates of both the anodic and cathodic flashes as well as the decay curve of the anodic glow.

As mentioned earlier, the intensity of the light emitted from all the metals except aluminum is so low that quantitative measurements are exceedingly difficult to obtain. Tantalum gives a fairly bright light initially, but soon a layer of oxide forms which virtually stops all current flow and the luminescence becomes very weak. Consequently, most measurements have been carried out with aluminum and aluminum alloys in

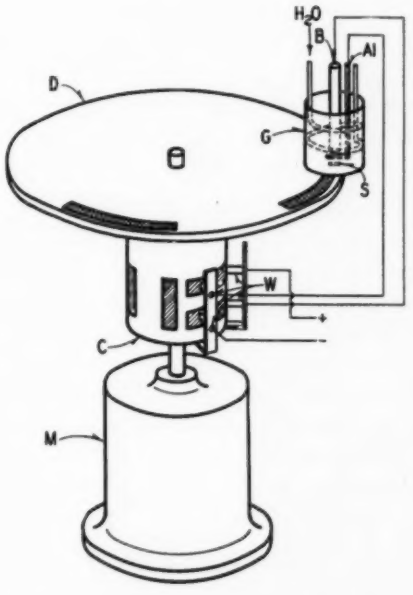

FIG. 2. Phosphoroscope for measuring decay times. Al, aluminum electrode; S, collimated slit; D, 15″ disk carrying films; C, commutator with graphite contacts at W; M, variable speed motor.

[7] F. Seitz, J. Chem. Phys. 6, 454 (1938).

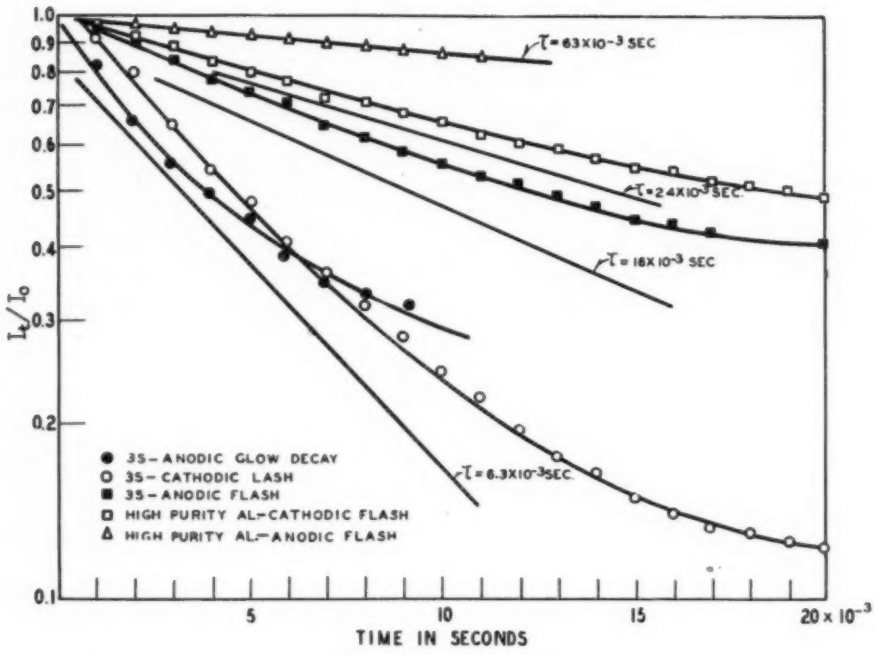

FIG. 3. Decay curves of anodic and cathodic flashes for high purity aluminum and aluminum with 1.2 percent added manganese (3S).

oxalic acid. Furthermore, as the temperature is elevated, the rate of acid attack upon the oxide layer becomes so excessive that the luminescence fades. Therefore, all measurements reported here were made at room temperature and below.

In general, the luminescence decays according to neither Eq. (1) nor (2). On the other hand, as can be seen from Fig. 3, the curves are either straight or concave upward when the logarithm of the ratio I_t/I_0 is plotted against the time (t). Here I_t is the intensity at time (t) measured from the time of maximum intensity (I_0) for the two flashes (with the current flowing) and from the time of shutting off the current for the anodic glow decay. (The maximum intensity is apparently reached as soon as the current starts to flow, at least within less than 10^{-3} second. This is quite different from ordinary fluorescence where the growth curve is determined by the same constants as the decay curve.) It is evident that the anodic flash decays more slowly than the cathodic flash, and the corresponding decays in the pure aluminum are much slower than for the alloy with 1.2 percent manganese added. In addition, it will be noticed that initially the decay of the anodic glow is approximately equal to that of the cathodic decay rate, but at longer times its curve is more nearly parallel to that for the anodic flash. The curves shown in Fig. 3 are typical. In general, the anodic flash decays approximately three times as slowly as the cathodic flash. Straight lines have been drawn to show the mean lifetime the flashes would have had if they had decayed exponentially.

The rate of the luminescent decay varies from one specimen to another, depending in an uncertain manner upon its treatment—applied voltage, current, temperature, solution concentration, etc. For example, in a series of measurements during which the anodic voltage varied by steps of ten from 50 to 110 volts, the lifetimes of both the anodic and cathodic flash decays showed no consistent relationship to voltage, current density, or any other variable measured. Behavior of this type results, in part, from the fact that voltage, current, acid concentration, and temperature are not independent variables, and small traces of impurities may greatly influence one or more of them. Further, slight changes in these factors often affect the composition of the oxide layer, such as water content and oxalate inclusions, etc. These in turn undoubtedly disturb

the radiating unit located in the oxide crystals by altering the potential field in which it is situated. For this reason attempted measurements of the variation of the lifetime with acid concentration and temperature failed to produce significant data.

From Fig. 1 it is evident that the cathodic flash was much brighter than the emission when the metal was made anodic. On the other hand, if one permits too long an interval to exist between disconnecting the metal from the anode and making the metal cathodic, the radiation will not be observed at all. The rate at which the electrode loses its ability to produce the cathodic flash has also been measured photographically. An apparatus was arranged so that a shutter could be operated to expose a spot on a film to the cathodic flash only. Usually several exposures are necessary to render one spot sufficiently dark for analysis. The dark time interval (time between being anodic and being made cathodic) was varied from one second to thirty-two seconds. This gave us a series of exposures on the film which were analyzed with the aid of a step plate and microphotometer.

The variation of the intensity of the cathodic flash with increasing dark time interval is shown in Fig. 4. The data exhibited were obtained with

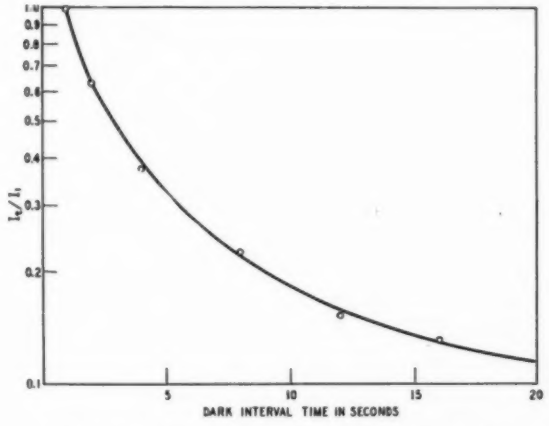

FIG. 4. Time variation of ability to produce the cathodic flash.

an electrode of aluminum containing 1.2 percent manganese in 0.25 percent oxalic acid at 21°C. The resultant curve has the same general shape as do those in Fig. 3. The mean lifetime of the ability to produce the cathodic flash is much longer than for the flash itself. In this case it is 22 seconds. Again the decay rate varies with acid concentration, etc., but not in a regular manner. The rate does seem to increase with temperature, though the results are uncertain at temperatures above 30°C.

The cathodic flash fails to appear if too long a dark interval elapses, but it can be secured twice in succession without making the electrode anodic if the first cathodic contact is not maintained long enough for the flash to decay completely. In other words, if the first contact lasts for approximately 0.001 second, we get a flash of light. A second contact will give another fainter emission, indicating that the condition for the cathodic radiation was not completely destroyed. Thus we see that the condition for radiation is destroyed by current flow, but the cathodic flash can be reproduced if the initial current does not flow too long. Once it is removed, the cathodic light will not appear again until after the electrode has been returned to the anodic polarity.

There are exceptions to this performance. When aluminum is used in extremely weak sulfuric acid solutions, the cathodic flash decays to a very low and virtually constant intensity. In such instances the condition for cathodic radiation is not destroyed entirely by the flow of charges.

IV. CURRENT-VOLTAGE LUMINESCENT INTENSITY RELATIONSHIPS

Oscillograph traces of the current and voltage were made concurrently with the decay studies. Drawings constructed from typical traces are reproduced in Figs. 5a and 5b. In Fig. 5a, which is the behavior when the metal is the anode, the point *A* indicates the time at which the impressed voltage was removed. Beyond this point the luminescent intensity was multiplied by a factor of 3.5 in order to represent it on this chart. It is to be understood, therefore, that the luminescence decays directly from *B* as soon as the impressed voltage is disconnected.

Obviously, there is no one to one correspondence between the voltage and the luminescent intensity or between the current and the radiation intensity. In fact, when the metal is cathodic

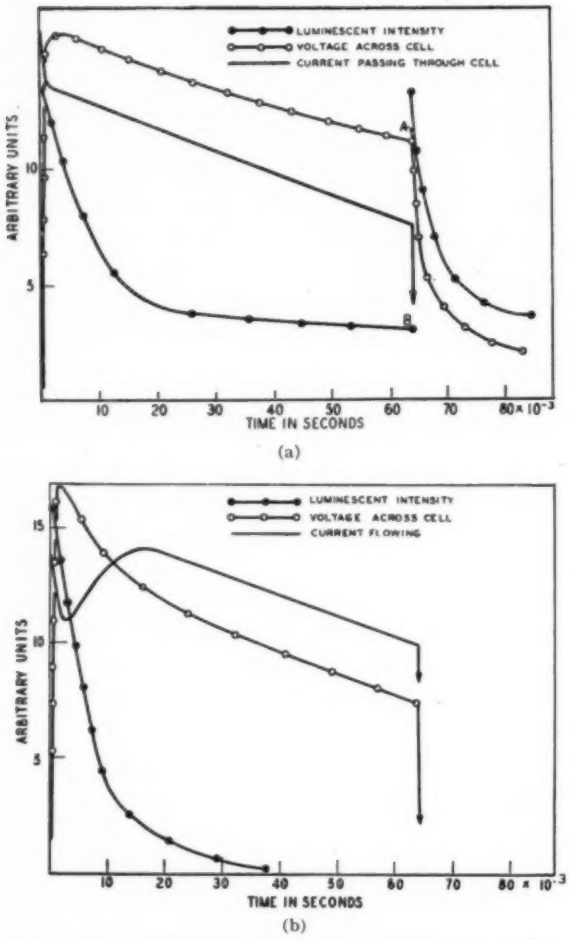

FIG. 5. (a) Variation with time of potential, current, and luminescent intensity when aluminum electrode is anodic. (b) Variation with time of potential, current, and luminescent intensity when aluminum electrode is cathodic.

(Fig. 5b), the high initial current decays quickly, passes through a minimum and then a secondary maximum before approaching a linear decay. While this is occurring, however, the luminescence is following a continuous decay. It is to be noticed that when the cathodic current is increasing, the voltage across the cell is decreasing more rapidly than it does during the remainder of the cycle. Therefore, the conductivity of the cell must be increasing rapidly during this time.

On the other hand, when the sheet of aluminum is first placed in the electrolyte, the luminescence does not become visible until the voltage reaches a certain threshold, this threshold varying from sample to sample.[3] At the same time, the oxide layer is increasing in thickness, thereby creating a higher resistance between the metal and electrolyte and necessitating a larger voltage to maintain the field within the oxide coating. The increase in emission, consequently, may be a function of the increase in the oxide layer thickness. In this case, the higher voltage merely indicates the coating's growth.

Another difficulty encountered in attempting to correlate the current with the radiation intensity is the fact that when the aluminum is the anode, almost all the current passing through the oxide is ionic.[8] Moreover, when the metal is the cathode, apparently all the current flowing through the layer is electronic.

V. SUMMARY OF LUMINESCENT PHENOMENA

The most important facts reported here and by others concerning the luminescence of valve metals in aqueous electrolytes may be summarized as follows:

1. When certain valve metals (aluminum, tungsten, tantalum, zinc, and magnesium) are made the anode in selected electrolytes (oxalic acid, etc.) and a current flows, light is emitted. The spectrum is characteristic of the metal and its impurities and is independent of the type of solution, if that solution is one which permits the formation of an oxide film upon the metal. For pure aluminum the spectral range is λλ3705–5875 with a maximum near λ4590. The maximum is shifted toward the red when higher voltages are applied.[5]

2. The luminescent intensity is not related directly to the current or to the voltage. Rather, it depends upon the specimen, its previous treatment, and other undefined conditions. The glow is brightest immediately after the metal has been the cathode in the cell. This bright flash has a mean decay time of the order of 2×10^{-2} second for aluminum containing 1.2 percent manganese when immersed in oxalic acid. The rate of decay varies with the metal. Furthermore, usually its decay curve is not a simple exponential function of the time but is concave upward when plotted as the logarithm of the intensity against time.

3. A momentary flash of light is emitted if the luminescing electrode is made the cathode im-

[8] J. D. Edwards and F. Keller, Trans. Electrochem. Soc. 79, 135 (1941).

mediately after having been the anode. The light decays in a manner similar to the anodic flash but at a rate approximately three times as great. In general, the light cannot be produced a second time until after the metal has been made anodic again. There are two exceptions, however. If the original cathodic contact is sufficiently short (about 1×10^{-3} second), a second contact will produce another weak flash of light. In addition, in certain solutions (e.g., extremely weak sulfuric acid), the cathodic light does not disappear but decays to a very low intensity, behaving as it does when made anodic.

4. The cathodic flash is much brighter than the anodic flash. The two flashes extend over the same spectra range but the cathodic flash may have a greater percent of its spectral energy in the red end of its spectrum, and both spectra are continuous.

5. If too long a dark interval elapses between removing the anodic polarity and making the electrode the cathode, the cathodic emission will not be observed. The ability to produce the cathodic flash decays in a manner similar to the two flashes of light but with a much longer decay time. Again the curve is either a straight line or is concave upward on a semi-log plot. The mean life is of the order of a minute but varies with the metal, electrolyte, etc. in an unpredictable manner (with the exception that for a given specimen the decay rate seems to increase with the temperature of the electrolyte).

VI. PROPOSED EXPLANATION

Since the luminescence is observed only with those metals whose oxides are excess semi-conductors, and since an intensive search has failed to reveal its presence when the electrode is a metal whose oxide is a defect semi-conductor, e.g. copper, nickel, and iron,[9] the indications are that the luminescence is connected in some way with the characteristic properties of the excess semi-conductor oxides. Excess semi-conductors are those which owe their conductivity either to an interstitial impurity or to a stoichiometric excess of the electropositive element which is assumed to be included within the interstices of the crystal.[10] In low fields the conductivity is due to electrons supplied by the impurity centers. The defect conductors are "hole" conductors, the conductivity resulting from a deficit of electrons. In barrier layer rectifiers the uni-linear conduction results from these peculiar conductivity properties of the oxide, hence it might be helpful to consider the mode of conduction as well as the theories concerning the uni-polarity occurring at semi-conductor metal contacts.

The modern theory of solids points out that the reason metals are good conductors and insulators are not lies in the fact that all the electronic energy levels in the lowest occupied band of the insulator are filled at 0°K and the gap between this band and the next one allowed is too great to be bridged by thermal energies. It can be shown[11] that lattice defects, interstitial atoms, and other disturbances create extra energy levels between the filled band and the empty conduction band. Thus the conduction in an excess semiconductor results from the donation of an electron by an impurity center to the conduction band, this center occupying a level not far below the band. When the electron is trapped in the neighborhood of the impurity center, it lies in a level just below the conduction band, and thermal excitation is possible. In defect semi-conductors an electron is trapped from the filled band into an impurity level lying immediately above the filled band. The hole left in the filled band carries a positive charge but is otherwise similar to an electron in an empty band. Therefore, the conduction is said to take place by these "positive holes."

Numerous theories of rectification[12] at semiconductor-metal contacts have been predicated upon the above basic ideas. All of these, at some point or other, fail to supply a complete and valid interpretation of the barrier layer valve action. During the past ten years, however, there has been developing a "space-charge" theory which

[9] J. Gundermann, K. Hauffe, and C. Wagner, Zeits. f. physik. Chemie **B37**, 148 (1937); H. von Baumbach and C. Wagner, Zeits. f. physik. Chemie **B24**, 59 (1934); C. Wagner and E. Koch, Zeits. f. physik. Chemie **B32**, 439 (1936).

[10] W. Hartmann, Zeits. f. Physik **102**, 709 (1936).

[11] N. F. Mott, and R. H. Gurney, *Electronic Processes in Ionic Crystals* (The Clarendon Press, 1940).

[12] N. F. Mott, Proc. Roy. Soc. **A171**, 27 (1939); B. Davydov, Tech. Phys. U.S.S.R. **5**, 87 (1938); A. H. Wilson, Proc. Roy. Soc. **A136**, 487 (1932); L. Nordheim, Zeits. f. Physik **75**, 434 (1932).

gives promise of supplying the solution. On the basis of this theory Schottky[13] has shown from qualitative considerations that the "barrier layer" is created by the passage of the current itself and need not depend for its existence upon changes in chemical composition. The barrier layer which is a positive space-charge at the metal semi-conductor contact alters its magnitude according to the direction of flow of the charge carriers. Thus, the uni-polar behavior exists.

The fundamentals of Schottky's formulation can be seen with the aid of Fig. 6. In the diagram n_1 represents the density of electrons in thermal equilibrium with the metal M_1, and similarly n_2 for M_2. The density of electrons associated with the thermally dissociable impurity centers in the semi-conductor (S–C) is indicated by $n_H(x)$. Schottky discusses a number of distribution functions of the impurity centers, but we shall consider that there is a linear concentration gradient decreasing from M_1 to M_2. Furthermore, it is assumed that $n_H(o) \gg n_1$, $n_H(L) \gg n_2$, and $n_1 \gg n_2$.

When thermal equilibrium is established, the product of the positive ions (n^+) and the negative ions (n^-) is constant. The exact value of one or the other may be determined by various factors, such as n_1 by the metal M_1. Therefore, when there is no current flowing, there is a positive space-charge created within the semi-conductor near each electrode.[14] When a potential is applied so that the electrons flow from M_2 toward M_1, then the positive space-charge near M_2 expands further into the semi-conductor, and the one near M_1 virtually disappears. In agreement with Poisson's equation, we have a high field intensity near M_2 and a lower intensity near M_1. When the potential is reversed, the space-charge near M_1 expands, and the other contracts. The expansion near M_1 is not as large as that previously before M_2. Consequently, since the dimension of the space-charge region determines the resistance of the circuit, a higher resistance will be experienced when the electrons flow from M_2 to the semi-conductor than when they flow in the opposite direction.

[13] W. Schottky, Zeits. f. Physik **113**, 367 (1939).

[14] Mott and Gurney (reference 11) show that, in general, when an excess semi-conductor is in contact with a metal, the former becomes positively charged near the boundary.

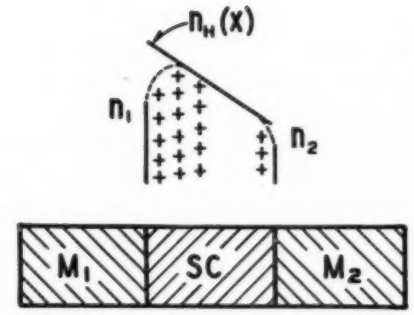

FIG. 6. Charge distribution within semi-conductor according to Schottky. M_1 and M_2, metals; SC, semi-conductor; n_1 and n_2, density of electrons in equilibrium with M_1 and M_2, respectively; $n_H(x)$, density of electrons in equilibrium with thermally dissociable impurity centers.

All the proposed theories of rectification (including Schottky's) are based upon the assumption that the interstitial impurity centers are fixed in position. On the contrary, recent studies of oxide growth[15] indicate that they have a rather high mobility. Hereafter, then, we shall assume that the interstitial impurity centers can migrate, their velocity depending upon the magnitude of the electric field present.

The processes within the oxide that produce the radiation are undoubtedly rather complex. Their clarification is further complicated by virtue of the fact that electrolytically formed aluminum oxide usually does not have a definite crystal structure.[8] The one notable exception is the film formed in a boric acid electrolyte in which case the structure of γ alumina is detected. Nevertheless, the luminescent effects are similar to those observed with oxalic acid, phosphoric acid, and sulfuric acid solutions. In the latter instances the films are amorphous. As mentioned before, however, electrolytically produced aluminum oxide is an excess semi-conductor.[10] Therefore, perhaps the least objectionable conjecture would be along the following lines.

When the aluminum sheet is made the anode, positive ions flow from the metal into the interstices of the oxide and establish a large positive ion space charge near the oxide-electrolyte interface as pictured by Schottky. (In Fig. 6 M_2 will represent the electrolyte.) Only a few electrons enter the oxide from the solution, but those that

[15] N. F. Mott, Trans. Faraday Soc. **36**, 472 (1940).

do enter will be able to create secondaries under the influence of the high field present which is of the order of 10^7 volts/cm. The secondaries are excited from the full band to the conduction band, leaving positive electron holes in the lower band. The electrons are then trapped by the positive ion impurity centers. The subsequent recombination of the electrons and positive holes gives rise to a visible radiation characteristic of the spacing between the impurity levels and the positive holes. Upon reversing the polarity, electrons flow out of the metal into the oxide and the impurity centers move back toward the metal. Light is emitted momentarily while vacant impurity centers remain to trap the secondary electrons. Once the ions are drawn out of the high field region, which is near the electrolyte, radiation ceases.

According to this proposal, the intensity of the luminescence will be dependent upon the number of free electrons flowing and the intensity of the field in which they find themselves. In order to determine the exact relationship existing between them and to correlate it with the details of the effect, it will be necessary to separate the electronic and ionic currents. So far, this has been impossible.

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The Resolution of Boundary Value Problems by Means of the Finite Fourier Transformation: General Vibration of a String

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The finite sine transformation and the finite cosine transformation are defined as the linear functional operations $\mathfrak{S}\{G\}=\int_0^\pi G(x)\sin nx dx=g_s(n)$ and $\mathfrak{C}\{G\}=\int_0^\pi G(x)\cos nx dx=g_c(n)$, respectively. The inversion of the product of two transforms can be made by means of four Faltung theorems. The finite sine transformation was applied to a boundary value problem of a general vibrating string, in which the partial differential equation has coefficients which may be functions of the time. A resolution was made of the transformed boundary value problem by the introduction of a fundamental set of solutions of the homogeneous transformed problem; an inversion in closed form was accomplished by the use of the Faltung theorems. A formal verification of this solution was made as well as a short survey of the applicability of the finite Fourier transformation to problems in engineering and physics.

I. INTRODUCTION

IN Part I of this paper some fundamental properties of the finite sine transformation and the finite cosine transformation are set forth. In Part II the finite sine transformation is applied to a problem in the general vibration of a string, and a resolution is effected by the introduction of a fundamental set of solutions of the homogeneous transformed problem. An inversion in closed form is accomplished by the use of Faltung theorems. In Part III a short survey is made of the applicability of the finite sine transformation

and the finite cosine transformation to problems in engineering and physics.[1]

1. Definition of the Transformation and Inversions

The finite sine transformation and the finite cosine transformation are defined by the linear functional operations

$$\mathfrak{S}\{V(x)\} = \int_0^\pi V(x) \sin nx dx = v_s(n),$$

and

$$\mathfrak{C}\{V(x)\} = \int_0^\pi V(x) \cos nx dx = v_c(n),$$

respectively. These coordinate the object function $V(x)$ to the result function,[2] or transform, $v(n)$, defined for integral arguments, for all $V(x)$ integrable in the fundamental interval. The following inversion formulas, justified according to the classical theorems of Fourier series, are used when a series solution is desired:

$$\mathfrak{S}^{-1}\{v_s\} \equiv \frac{2}{\pi} \sum_{n=1}^{\infty} v_s(n) \sin nx = V(x),$$

and

$$\mathfrak{C}^{-1}\{v_c\} \equiv \frac{v_c(0)}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} v_c(n) \cos nx = V(x).$$

However, if a non-series solution is desirable, that is, in "closed form," the Faltung theorems are used.

2. Faltung Theorems

The Faltung or composition of two functions V_1 and V_2 is defined as follows:[1(b)]

Definition. If $V_1(x)$, $(-2\pi \leq x \leq 2\pi)$, and $V_2(x)$, $(-\pi \leq x \leq \pi)$ are bounded and integrable, then the Faltung of $V_1(x)$ and $V_2(x)$ is defined as

$$V_1 * V_2 \equiv \int_{-\pi}^{\pi} V_1(x-\xi) V_2(\xi) d\xi.$$

In order to carry out the inversions of result functions expressed as products, we make use of the fact that the Faltung of two even functions is again an even function, the Faltung of two odd functions is also an even function, and the Faltung of an odd function and an even function is an odd function.

Hans Kniez[1c] states the following four Faltung theorems:

Theorem I. If V_1 is odd and periodic with period 2π, and if V_2 is even, then

$$\mathfrak{S}\{\tfrac{1}{2} V_1 * V_2\} = \mathfrak{S}\{V_1\}\mathfrak{C}\{V_2\}.$$

Theorem II. If V_1 and V_2 are even functions and V_1 is periodic with period 2π, then

$$\mathfrak{C}\{\tfrac{1}{2} V_1 * V_2\} = \mathfrak{C}\{V_1\}\mathfrak{C}\{V_2\}.$$

Theorem III. If V_1 and V_2 are odd functions and V_1 is periodic with period 2π, then

$$\mathfrak{C}\{\tfrac{1}{2} V_1 * V_2\} = -\mathfrak{S}\{V_1\}\mathfrak{S}\{V_2\}.$$

Theorem IV. If V_1 is even and periodic with period 2π, and if V_2 is odd, then

$$\mathfrak{S}\{\tfrac{1}{2} V_1 * V_2\} = \mathfrak{C}\{V_1\}\mathfrak{S}\{V_2\}.$$

In these Faltung theorems, if both of the functions V_1 and V_2 are periodic with period 2π, we can commute V_1 and V_2 in the Faltung $V_1 * V_2$.

3. Sine Transforms[3]

(a) $$\mathfrak{S}\left\{\frac{\pi - x}{\pi}\right\} = \frac{1}{n},$$

(b) $$\mathfrak{S}\left\{\frac{x}{\pi}\right\} = \frac{(-1)^{n+1}}{n}.$$

[1] For a more complete discussion, see (a) Herbert K. Brown, "Resolution of boundary value problems by means of the finite Fourier transformation: two general heat flow problems," Summary contained in Bull. Am. Math. Soc. **48**, 522 (1942) (Abstract 246); (b) G. Doetsch, "Integration von Differentialgleichungen vermittels der endlichen Fourier Transformation," Math. Ann. **112** 52–68 (1935); (c) Hans Kniez, "Lösung von Randwertproblemen bei Systemen gewöhnlicher Differentialgleichungen vermittels der endlichen Fourier Transformation," Math. Zeits. **44**, 266–291 (1938).

[2] The lower case letters shall be used to signify the transforms of the functions designated by the corresponding capital letters. Hereafter, instead of using the symbols $v_s(n)$ and $v_c(n)$ to designate the sine and cosine transforms, respectively, we shall use the symbol $v(n)$ for both, whenever it is evident which one is meant.

[3] E. P. Adams, *Smithsonian Mathematical Formulas and Tables of Elliptic Functions* (Smithsonian Institution, Washington, 1939), p. 138.

(c) $\mathfrak{S}\{V(\pi-x)\}=(-1)^{n+1}\mathfrak{S}\{V(x)\}$.

(d) $\mathfrak{S}\{V'(x)\}=\int_0^\pi V'(x)\sin nx\,dx=-n\mathfrak{C}\{V(x)\}$.

(e) $\mathfrak{S}\{V''(x)\}=-n[(-1)^nV(\pi)-V(0)]$
$-n^2\mathfrak{S}\{V(x)\}=-n^2\mathfrak{S}\{V(x)\}$

if $V(\pi)=V(0)=0$.

(f) $\mathfrak{S}\{V^{(2\alpha)}(x)\}=-n[(-1)^nV^{(2\alpha-2)}(\pi)$
$-V^{(2\alpha-2)}(0)]+n^3[(-1)^nV^{(2\alpha-4)}(\pi)$
$-V^{(2\alpha-4)}(0)]-\cdots+(-1)^\alpha n^{2\alpha-1}$
$\times[(-1)^nV(\pi)-V(0)]+(-n^2)^\alpha\mathfrak{S}\{V(x)\}$.

4. Cosine Transforms

(a) $\mathfrak{C}\left\{\frac{\pi}{3}-x+\frac{x^2}{2\pi}\right\}=\frac{1}{n^2}$,

(b) $\mathfrak{C}\left\{-\frac{\pi}{6}+\frac{x^2}{2\pi}\right\}=\frac{(-1)^{n+1}}{n^2}$.

(c) $\mathfrak{C}\{K\}=0 \quad (n>0)$,
$=\frac{\pi K}{2} \quad (n=0) \quad (K=\text{constant})$.

(d) $\mathfrak{C}\{V(\pi-x)\}=(-1)^n\mathfrak{C}\{V(x)\}$.

(e) $\mathfrak{C}\{V'(x)\}=\int_0^\pi V'(x)\cos nx\,dx=[(-1)^nV(\pi)$
$-V(0)]+n\mathfrak{S}\{V(x)\}=n\mathfrak{S}\{V(x)\}$

if $V(\pi)=V(0)=0$.

(f) $\mathfrak{C}\{V''(x)\}=[(-1)^nV'(\pi)-V'(0)]$
$-n^2\mathfrak{C}\{V(x)\}=-n^2\mathfrak{C}\{V(x)\}$

if $V'(\pi)=V'(0)=0$.

(g) $\mathfrak{C}\{V^{(2\alpha)}(x)\}=[(-1)^nV^{(2\alpha-1)}(\pi)-V^{(2\alpha-1)}(0)]$
$-n^2[(-1)^nV^{(2\alpha-3)}(\pi)-V^{(2\alpha-3)}(0)]+\cdots$
$+(-1)^{\alpha-1}n^{2\alpha-2}[(-1)^nV'(\pi)$
$-V'(0)]+(-n^2)^\alpha\mathfrak{C}\{V(x)\}$.

II. GENERAL VIBRATION OF A STRING

The application of the finite sine transformation or the finite cosine transformation to certain boundary value problems reduces them to problems in ordinary differential equations. If the transformed equation is linear and of the first order then the problem can be solved directly and an effective resolution in terms of a much simpler problem is possible.[1a] However, if the transformed equation is of the second order, as in the boundary value problem now to be considered, in general it cannot be solved directly. Nevertheless, one can solve such problems indirectly in terms of a fundamental set of solutions of the homogeneous transformed ordinary differential equation.

It is to be understood that the presentation given here is heuristic, and that the verifications are intended merely as formal checks.

1. Physical Description and Mathematical Interpretation

A stretched string of length π is immersed in a fluid whose viscosity, which may be a function of the time, sets up a resistance to transverse vibrations proportional to the displacement. The modulus of elasticity of the string may depend upon the length of time of immersion in the fluid. Not only is the string as a whole subjected to external forces parallel to the direction of displacement, but the endpoints also are given preassigned displacements. The initial configuration of the string and the initial velocity of points along the string are given likewise by preassigned functions.

The transverse displacements $V(x, t)$ of such a string have the following properties in the region $R: 0<x<\pi,\ t>0$:

(a) $L(V)\equiv V_{tt}-C_1(t)V_{xx}+C_2(t)V(x,t)$
$=P(x,t)$ in R,

(b)[4] $V(+0,t)=C_3(t),\ V(\pi-0,t)=C_4(t)$ $(t>0)$, (1)

(c) $V(x,+0)=F_1(x),\ V_t(x,+0)=F_2(x)$ $(0<x<\pi)$,

[4] $\lim_{x\to 0}V(x,t)=V(+0,t)$.

where the literal subscripts denote partial derivatives.

For the purpose of abbreviating the forms that enter into our work, we shall assume that the functions in this description satisfy the following conditions:

(a) F_1, F_2, P and their first three derivatives with respect to x vanish at the endpoints of the fundamental interval $0 \leq x \leq \pi$. This condition guarantees the continuity in x of the odd, periodic extensions of the functions F_1, F_2, P and their first three derivatives for all x. (2)

(b) Since we do not want the string to break at the endpoints when t approaches zero, condition (a) demands that

$$C_3(0) = C_4(0) = C_3'(0) = C_4'(0) = 0.$$

Since we are not going to give a complete check of our work, we cannot state what further conditions on these functions are sufficient to establish the solution of our problem.

2. Method of Superposition and Reduction

It is obvious that the sum

$$V(x, t) = \sum_{i=1}^{5} {}_iV(x, t)$$

satisfies all the conditions of (II—1.1),[5] where ${}_iV(x, t)$ $(i = 1, 2, 3, 4, 5)$ are the solutions of the following five problems:

Problem A

(a) $L({}_1V) = P(x, t)$ in R,
(b)[6] $M({}_1V) = 0$ $(t > 0)$,
(c)[7] $N({}_1V) = 0$ $(0 < x < \pi)$.

Problem B

(a) $L({}_2V) = 0$ in R,
(b) ${}_2V(+0, t) = C_3(t)$, ${}_2V(\pi - 0, t) = 0$ $(t > 0)$,
(c) $N({}_2V) = 0$ $(0 < x < \pi)$.

Problem C

(a) $L({}_3V) = 0$ in R,
(b) ${}_3V(+0, t) = 0$, ${}_3V(\pi - 0, t) = C_4(t)$ $(t > 0)$,
(c) $N({}_3V) = 0$ $(0 < x < \pi)$.

Problem D

(a) $L({}_4V) = 0$ in R,
(b) $M({}_4V) = 0$ $(t > 0)$,
(c) ${}_4V(x, +0) = F_1(x)$, ${}_4V_t(x, +0) = 0$ $(0 < x < \pi)$.

Problem E

(a) $L({}_5V) = 0$ in R,
(b) $M({}_5V) = 0$ $(t > 0)$,
(c) ${}_5V(x, +0) = 0$, ${}_5V_t(x, +0) = F_2(x)$ $(0 < x < \pi)$.

Note. If in Problem B we replace $C_3(t)$ by $C_4(t)$ and x by $\pi - x$, then ${}_2V(x, t)$ reduces to ${}_3V(x, t)$. For this reason we shall not have to consider Problem C as a problem to be solved.

We return to the original form (II—1.1) of our problem in order to make a formal application of the finite sine transformation. Setting the function $C_4(t)$ identically zero, for the time being, the result of the transformation is a problem in ordinary differential equations of the second order in t:

$$\begin{aligned} &\text{(a)}\ \mathfrak{L}(v) \equiv v''(n, t) + [C_2(t) + n^2 C_1(t)]v(n, t) \\ &\qquad = q(n, t), \\ &\text{(b)}\ v(n, 0) = f_1(n),\ v'(n, 0) = f_2(n), \end{aligned} \tag{1}$$

where $q(n, t) \equiv p(n, t) + nC_3(t)C_1(t)$.

Since our equation is of the second order, it cannot, in general, be solved directly. But by applying the theory of ordinary differential equations of the second order,[8] in particular the concept of a fundamental set of solutions, we can solve for $v(n, t)$ indirectly.

Any linearly independent set of two solutions $v_1(n, t)$ and $v_2(n, t)$ of the linear homogeneous ordinary differential equation of the second order,

$$p_0(t)\frac{d^2v}{dt^2} + p_1(t)\frac{dv}{dt} + p_2(t)v = 0,$$

is said to form a *fundamental set* or *fundamental*

[5] (II—1.1) refers to the item (1) in the first section of II.
[6] The expression $M({}_1V) = 0$ means that ${}_1V$ vanishes at both endpoints of the interval $(0, \pi)$.
[7] The expression $N({}_1V) = 0$ means that the initial values of ${}_1V(x, t)$ and ${}_1V_t(x, t)$ are both zero.
[8] E. L. Ince, *Ordinary Differential Equations* (Longmans, Green, London, 1927), p. 119.

system. This is equivalent to saying that the Wronskian[9] $\Delta(v_1, v_2)$ does not vanish in the open interval $(0<t<T)$. The general solution of the homogeneous equation can be written

$$v=k_1v_1+k_2v_2.$$

In the case of the non-homogeneous transformed problem in (II—2.1), we can select our fundamental set as the solutions of

$$\begin{array}{ll} \text{(a) } \mathfrak{L}(v_1)=0, & \text{(a) } \mathfrak{L}(v_2)=0, \\ \text{(b) } v_1(n,0)=0, & \text{(b) } v_2(n,0)=1/n, \\ \quad v_1'(n,0)=1/n, & \quad v_2'(n,0)=0. \end{array} \tag{2}$$

We shall call $v_1(n, t)$ *and* $v_2(n, t)$ *transformed auxiliary functions. They correspond to the auxiliary functions* $V_1(x, t)$ *and* $V_2(x, t)$, which we shall define in the next section.

We can express $v(n, t)$ in terms of the transformed auxiliary functions $v_1(n, t)$ and $v_2(n, t)$ by Lagrange's method of variation of parameters of the general solution: $v=k_1v_1+k_2v_2$. The result is

$$v(n,t)=f_1(n)nv_2(n,t)+f_2(n)nv_1(n,t) + \int_0^t q(n,\tau)\frac{w(n,t,\tau)}{\Delta(v_1,v_2)}d\tau, \tag{3}$$

where $w(n, t, \tau)=v_1(n, t)v_2(n, \tau)-v_1(n, \tau)v_2(n, t)$, and the Wronskian $\Delta(v_1, v_2)=1/n^2$.

When we substitute the expanded form of $q(n, t)$ in (3) we see that

$$v(n,t)=\sum_{i=1,\ i\neq 3}^{5} {}_iv(n,t),$$

where the ${}_iv(n, t)$ $(i=1, 2, 4, 5)$ are the transforms of the ${}_iV(x, t)$. These transforms can be expressed as follow:

$$\begin{aligned} {}_1v(n,t)&=\int_0^t p(n,\tau)n^2w(n,t,\tau)d\tau, \\ {}_2v(n,t)&=\int_0^t C_3(\tau)C_1(\tau)n^2w(n,t,\tau)d\tau, \\ {}_4v(n,t)&=f_1(n)nv_2(n,t), \\ {}_5v(n,t)&=f_2(n)nv_1(n,t). \end{aligned} \tag{4}$$

[9] $\Delta(v_1, v_2)\equiv[v_1'(n, t)v_2(n, t)-v_1(n, t)v_2'(n, t)]$.

3. Inversion in Terms of Auxiliary Functions

In expressing $v(n, t)$ as the solution of (II—2.1) we made use of the transformed auxiliary functions $v_1(n, t)$ and $v_2(n, t)$. The corresponding object functions $V_1(x, t)$ and $V_2(x, t)$ have the following properties in the region $R: 0<x<\pi, t>0$:

$$\begin{aligned} &\text{(a)} \quad L(V_1)=0 && \text{in } R, \\ &\text{(b)} \quad M(V_1)=0 && (t>0), \\ &\text{(c)} \quad V_1(x,+0)=0, \\ &\qquad V_{1t}(x,+0)=\frac{\pi-x}{\pi} && (0<x<\pi). \end{aligned} \tag{1}$$

$$\begin{aligned} &\text{(a)} \quad L(V_2)=0 && \text{in } R, \\ &\text{(b)} \quad M(V_2)=0 && (t>0), \\ &\text{(c)} \quad V_2(x,+0)=\frac{\pi-x}{\pi}, \\ &\qquad V_{2t}(x,+0)=0 && (0<x<\pi). \end{aligned} \tag{2}$$

Note. Instead of using the function $(\pi-x)/\pi$ in the initial conditions we could use $k(\pi-x)/\pi$, where k is a constant which can be made as small as desired. This would make it possible to put a bound to the discontinuity of the auxiliary functions at the endpoints of the string as t approaches zero, and thus bring the mathematical interpretation into closer accordance with the physical description.

The inversion of $v(n, t)$ depends upon the inversions of the ${}_iv(n, t)$ of which ${}_1v(n, t)$ and ${}_2v(n, t)$ are expressed in terms of $w(n, t, \tau)$. The following inversions of forms of $w(n, t, \tau)$ are given for future reference:

$$\begin{aligned} w(n,t,\tau)&=v_1(n,t)v_2(n,\tau)-v_1(n,\tau)v_2(n,t) \\ &=\mathfrak{S}\{V_1(x,t)\}\mathfrak{S}\{V_2(x,\tau)\} \\ &\quad -\mathfrak{S}\{V_1(x,\tau)\}\mathfrak{S}\{V_2(x,t)\} \\ &=\mathfrak{C}\{-\tfrac{1}{2}[V_1(x,t)*V_2(x,\tau) \\ &\quad -V_1(x,\tau)*V_2(x,t)]\}. \end{aligned}$$

We use Faltung theorem III to make the *definition:*

$$w(n,t,\tau)=\mathfrak{C}\{W(x,t,\tau)\}, \tag{3}$$

where

$$W(x,t,\tau)=-\int_{-\pi}^{\pi}[V_1(x-\xi,t)V_2(\xi,\tau)-V_1(x-\xi,\tau)V_2(\xi,t)]d\xi,$$

where both V_1 and V_2 are defined as the odd and periodic extensions of the original functions V_1 and V_2. Since both V_1 and V_2 are periodic with period 2π we can commute them in the Faltung $V_1 * V_2$.

From the Wronskian $\Delta(v_1, v_2)=1/n^2$ we have

$$\frac{1}{n}=n\Delta(v_1,v_2)=n\lim_{\tau\to t}w_t(n,t,\tau)$$

$$=\lim_{\tau\to t}\frac{\partial}{\partial t}[n\mathfrak{C}\{W(x,t,\tau)\}]$$

$$=\lim_{\tau\to t}\frac{\partial}{\partial t}[-\mathfrak{S}\{W_x(x,t,\tau)\}]=-\mathfrak{S}\{\lim_{\tau\to t}W_{xt}\}.$$

So that we have the heuristic formula,

$$\frac{\pi-x}{\pi}=-\lim_{\tau\to t}W_{xt}=\lim_{\tau\to t}W_{x\tau}. \tag{4}$$

Furthermore it can be shown that

$$\lim_{\tau\to 0}W_{x\tau}=V_2(x,t).$$

We can make a composition of the Wronskian $\Delta(v_1, v_2)$ with an arbitrary function as follows:

$$g(n,t)=g(n,t)n^2\lim_{\tau\to t}w_t(n,t,\tau)$$

$$=g(n,t)\lim_{\tau\to t}\frac{\partial}{\partial t}[n^2w(n,t,\tau)]$$

$$=g(n,t)\lim_{\tau\to t}\frac{\partial}{\partial t}[\mathfrak{C}\{-W_{xx}(x,t,\tau)\}]$$

$$=-\mathfrak{S}\{G(x,t)\}\mathfrak{C}\{\lim_{\tau\to t}W_{xxt}(x,t,\tau)\}$$

$$=\mathfrak{S}\{-\tfrac{1}{2}\lim_{\tau\to t}[G(x,\tau)*W_{xxt}(x,t,\tau)]\}.$$

Therefore, assuming that we can make the necessary interchanges of operations used in this formulation, we have

$$G(x,t)=-\tfrac{1}{2}\lim_{\tau\to t}\int_{-\pi}^{\pi}G(x-\xi,\tau)W_{\xi\xi t}(\xi,t,\tau)d\xi, \tag{5}$$

where G is odd and periodic, and W_{xxt} is even.

By simple manipulations we have the following formal results:

$$W_x(+0,t,\tau)=W_x(\pi-0,t,\tau)=0,$$

$$\lim_{\tau\to t}W_x=0,\quad \lim_{\tau\to 0}W_x=-V_1(x,t). \tag{6}$$

$$W(x,t,t)=W_x(x,t,t)=W_{xx}(x,t,t)=0. \tag{7}$$

$$n^2w(n,t,\tau)=-\mathfrak{C}\{W_{xx}(x,t,\tau)\}. \tag{8}$$

We can now perform the inversions of the solutions ${}_iv(n,t)$ in (II—2.4), using the inversion formulas developed above.

Consider the inversion of ${}_1v(n,t)$:

$${}_1v(n,t)=\int_0^t p(n,\tau)n^2w(n,t,\tau)d\tau$$

$$=\int_0^t\mathfrak{S}\{P(x,\tau)\}\mathfrak{C}\{-W_{xx}(x,t,\tau)\}d\tau$$

$$=-\int_0^t\mathfrak{S}\{\tfrac{1}{2}P(x,\tau)*W_{xx}(x,t,\tau)\}d\tau.$$

We have then

$${}_1V(x,t)=-\tfrac{1}{2}\int_0^t\int_{-\pi}^{\pi}P(x-\xi,\tau)W_{\xi\xi}(\xi,t,\tau)d\xi d\tau, \tag{9}$$

where P is odd and periodic, and $W_{\xi\xi}$ is even.

Consider the inversion of ${}_2v(n,t)$:

$${}_2v(n,t)=\int_0^t C_3(\tau)C_1(\tau)n^2w(n,t,\tau)d\tau$$

$$=-\int_0^t C_3(\tau)C_1(\tau)n\mathfrak{C}\{W_{xx}(x,t,\tau)\}d\tau$$

$$=\int_0^t C_3(\tau)C_1(\tau)\mathfrak{S}\{W_{xxx}(x,t,\tau)\}d\tau.$$

Therefore, assuming that we can interchange the order of the integration and the sine transformation, we have

$${}_2V(x,t)=\int_0^t C_3(\tau)C_1(\tau)W_{xxx}(x,t,\tau)d\tau. \tag{10}$$

There are still the two forms ${}_4v(n,t)$ and ${}_6v(n,t)$ to be considered; it is obvious, however,

that their inversions will be similar. Consider the expression ${}_4v(n, t)$:

$$ {}_4v(n,t) = f_1(n) n v_2(n,t) = \mathfrak{S}\{F_1(x)\}\mathfrak{C}\{V_{2x}(x,t)\} $$
$$ = \mathfrak{S}\{\tfrac{1}{2}F_1(x) * V_{2x}(x,t)\}. $$

By Faltung theorem I,

$$ {}_4V(x,t) = \tfrac{1}{2}\int_{-\pi}^{\pi} F_1(x-\xi)\, V_{2\xi}(\xi,t)\, d\xi, \qquad (11) $$

$$ {}_5V(x,t) = \tfrac{1}{2}\int_{-\pi}^{\pi} F_2(x-\xi)\, V_{1\xi}(\xi,t)\, d\xi, \qquad (12) $$

where F_1 and F_2 are odd and periodic, and $V_{1\xi}$ and $V_{2\xi}$ are even.

4. Formal Verification of the Solutions ${}_iV(x, t)$

We shall now give a formal check of the solutions ${}_iV(x, t)$ $(i=1, 2, 4, 5)$ of the problems A, B, D, and E. We shall consider each in turn.

Problem A

(a) $L({}_1V) = P(x,t)$ in R,

(b) $M({}_1V) = 0$ $(t>0)$,

(c) $N({}_1V) = 0$ $(0<x<\pi)$.

According to (II—3.9),

$$ {}_1V(x,t) = -\tfrac{1}{2}\int_0^t\int_{-\pi}^{\pi} P(x-\xi,\tau)\, W_{\xi\xi}(\xi,t,\tau)\, d\xi d\tau, \quad (1) $$

where P is odd and periodic, and $W_{\xi\xi}$ is even.

As t approaches zero we see that ${}_1V(x, +0) = 0$. In order to check the endpoint conditions we integrate (1) by parts with respect to ξ and find that

$$ {}_1V(x,t) = -\tfrac{1}{2}\int_0^t\int_0^{\pi} [P_x(x-\xi,\tau) - P_x(x+\xi,\tau)] W_\xi(\xi,t,\tau)\, d\xi d\tau, \quad (2) $$

since P is odd and periodic, and $W_\xi = 0$ at the endpoints of the interval $(0, \pi)$. Now let x approach zero and π. The result is ${}_1V(+0, t) = 0$ and ${}_1V(\pi-0, t) = 0$, since $P_x(\xi,\tau) = P_x(-\xi,\tau)$ and $P_x(\pi+\xi,\tau) = P_x(-\pi-\xi,\tau) = P_x(\pi-\xi,\tau)$.

If we integrate by parts twice in (1), we obtain an alternative form of ${}_1V(x, t)$, namely,

$$ {}_1V(x,t) = -\tfrac{1}{2}\int_0^t\int_{-\pi}^{\pi} P_{xx}(x-\xi,\tau)\, W(\xi,t,\tau)\, d\xi d\tau. \quad (3) $$

To check the initial velocity condition and the partial differential equation, we need the derivative of ${}_1V(x, t)$ with respect to t. From (1) and (3) we find formally by Leibnitz' rule that

$$ {}_1V_t(x,t) = -\tfrac{1}{2}\int_0^t\int_{-\pi}^{\pi} P(x-\xi,\tau)\, W_{\xi\xi t}(\xi,t,\tau)\, d\xi d\tau, $$

or else

$$ {}_1V_t(x,t) = -\tfrac{1}{2}\int_0^t\int_{-\pi}^{\pi} P_{xx}(x-\xi,\tau)\, W_t(\xi,t,\tau)\, d\xi d\tau. $$

From these forms it follows that ${}_1V_t(x, +0) = 0$. The second derivative of ${}_1V(x, t)$ with respect to t can be expressed as follows:

$$ {}_1V_{tt}(x,t) = -\tfrac{1}{2}\lim_{\tau\to t}\int_{-\pi}^{\pi} P(x-\xi,\tau)\, W_{\xi\xi t}(\xi,t,\tau)\, d\xi $$
$$ -\tfrac{1}{2}\int_0^t\int_{-\pi}^{\pi} P_{xx}(x-\xi,\tau)\, W_{tt}(\xi,t,\tau)\, d\xi d\tau. \quad (4) $$

We shall now show that $W(x, t, \tau)$ satisfies the partial differential equation of the auxiliary problems (II—3.1). We have from (II—3.3) that

$$ W(x,t,\tau) = -\int_{-\pi}^{\pi} [V_1(x-\xi,t)\, V_2(\xi,\tau) - V_1(x-\xi,\tau)\, V_2(\xi,t)]\, d\xi, $$
$$ = -\int_{-\pi}^{\pi} [V_1(x-\xi,t)\, V_2(\xi,\tau)]\, d\xi + \int_{-\pi}^{\pi} [V_1(\xi,\tau)\, V_2(x-\xi,t)]\, d\xi, $$

since both V_1 and V_2 are periodic with period 2π (see I—5).

Since the partial differential equation of the auxiliary problems is linear and homogeneous, it follows that

$$ L(W) = -\int_{-\pi}^{\pi} L(V_1)\, V_2\, d\xi + \int_{-\pi}^{\pi} V_1 L(V_2)\, d\xi = 0 \qquad (t>0),\ (0<x<\pi). $$

That is to say,

$$W_{tt}(x,t,\tau)=C_1(t)W_{xx}(x,t,\tau)-C_2(t)W(x,t,\tau). \quad (5)$$

Substituting this result into (3), and using the development from (II—3.5), that

$$P(x,t)=-\tfrac{1}{2}\lim_{\tau\to t}\int_{-\pi}^{\pi}P(x-\xi,\tau)W_{\xi\xi t}(\xi,t,\tau)d\xi,$$

we find that

$$_1V_{tt}(x,t)=P(x,t)$$
$$-C_1(t)\tfrac{1}{2}\int_0^t\int_{-\pi}^{\pi}P_{xx}(x-\xi,\tau)W_{\xi\xi}(\xi,t,\tau)d\xi d\tau$$
$$+C_2(t)\tfrac{1}{2}\int_0^t\int_{-\pi}^{\pi}P_{xx}(x-\xi,\tau)W(\xi,t,\tau)d\xi d\tau.$$

Hence from (1) and (3) it follows formally that

$$_1V_{tt}(x,t)=P(x,t)+C_1(t)\,{}_1V_{xx}-C_2(t)\,{}_1V \quad \text{in } R.$$

This concludes the verification of problem A.

Problem B

(a) $L(_2V)=0$ in R,

(b) $_2V(+0,t)=C_3(t)$, $_2V(\pi-0,t)=0$ $(t>0)$,

(c) $N(_2V)=0$ $(0<x<\pi)$.

According to (II—3.10),

$$_2V(x,t)=\int_0^t C_3(\tau)C_1(\tau)W_{xxx}(x,t,\tau)d\tau. \quad (6)$$

In (5) we showed that $W(x,t,\tau)$ satisfies the partial differential equation, $L(W)=0$. We use this result in (6), and find with the use of (II—3.4) and (II—3.6) that

$$_2V(x,t)=\int_0^t C_3(\tau)[W_{x\tau\tau}+C_2(\tau)W_x]d\tau$$
$$=C_3(t)\lim_{\tau\to t}W_{x\tau}-C_3'(t)\lim_{\tau\to t}W_x$$
$$-C_3(0)\lim_{\tau\to 0}W_{x\tau}+C_3'(0)\lim_{\tau\to 0}W_x$$
$$+\int_0^t[C_3''(\tau)+C_2(\tau)C_3(\tau)]W_x d\tau. \quad (7)$$

$$_2V(x,t)=C_3(t)\left(\frac{\pi-x}{\pi}\right)$$
$$+\int_0^t[C_3''(\tau)+C_2(\tau)C_3(\tau)]W_x(x,t,\tau)d\tau,$$

since $C_3(0)=C_3'(0)=0$. Since

$$W_x(+0,t,\tau)=W_x(\pi-0,t,\tau)=0,$$

for $t>0$, and $C_3(0)=0$, we have that

$$_2V(+0,t)=C_3(t), \quad {}_2V(\pi-0,t)={}_2V(x,+0)=0.$$

In (7) we take the derivative with respect to t, and find that

$$_2V_t(x,t)=C_3'(t)\left(\frac{\pi-x}{\pi}\right)+\int_0^t[C_3''+C_2C_3]W_{xt}d\tau,$$

since $W_x(x,t,t)=0$. From this form we see that the initial velocity is zero, i.e., $_2V_t(x,+0)=0$.

We now check the partial differential equation $L(_2V)=0$:

$$_2V_{tt}(x,t)=C_3''(t)\left(\frac{\pi-x}{\pi}\right)$$
$$-[C_3''(t)+C_2(t)C_3(t)]\left(\frac{\pi-x}{\pi}\right)$$
$$+\int_0^t[C_3''(\tau)+C_2(\tau)C_3(\tau)]W_{xtt}(x,t,\tau)d\tau,$$

since

$$\lim_{\tau\to t}W_{xt}=-(\pi-x)/\pi.$$

That is, for $t>0$ and $0<x<\pi$,

$$_2V_{tt}(x,t)=-C_2(t)C_3(t)\left(\frac{\pi-x}{\pi}\right)$$
$$+\int_0^t[C_3''+C_2C_3][C_1(t)W_{xxx}$$
$$-C_2(t)W_x]d\tau=-C_2(t)\left\{C_3(t)\left(\frac{\pi-x}{\pi}\right)\right.$$
$$\left.+\int_0^t[C_3''+C_2C_3]W_x d\tau\right\}+C_1(t)\,{}_2V_{xx}.$$

It therefore follows that

$$ {}_2V_{tt}(x,t) = -C_2(t)\,{}_2V(x,t) + C_1(t)\,{}_2V_{xx} \quad \text{in } R. $$

Note. This formal proof can be carried out without having $C_3(0) = C_3'(0) = 0$, using of course additional terms in the form of ${}_2V(x,t)$ in the successive steps of the verification (for instance, see the form of ${}_2V(x,t)$ as given in (II—4.7)). In checking the initial condition, for example, it is the case that

$$ {}_2V(x,+0) = C_3(0)\left(\frac{\pi - x}{\pi}\right) - C_3(0)\,V_2(x,+0) = 0, $$

since

$$ V_2(x,+0) = (\pi - x)/\pi. $$

Problem D

(a) $L({}_4V) = 0$ in R,

(b) $M({}_4V) = 0$ $(t > 0)$,

(c) ${}_4V(x,+0) = F_1(x)$, ${}_4V_t(x,+0) = 0$ $(0 < x < \pi)$.

According to (II—3.11),

$$ {}_4V(x,t) = \tfrac{1}{2}\int_{-\pi}^{\pi} F_1(x-\xi)\,V_{2\xi}(\xi,t)\,d\xi, \qquad (8) $$

where F_1 is odd and periodic, and $V_{2\xi}$ is even. When we restrict ξ to the interval $(0, \pi)$ then our solution can be written as

$$ {}_4V(x,t) = \tfrac{1}{2}\int_0^{\pi} [F_{1x}(x-\xi) - F_{1x}(x+\xi)]\,V_2(\xi,t)\,d\xi. \qquad (9) $$

Since F_1 is odd and periodic with period 2π it follows formally that ${}_4V(+0,t) = {}_4V(\pi - 0, t) = 0$ for $t > 0$. Further, we see that

$$ \begin{aligned} {}_4V(x,+0) &= \tfrac{1}{2}\int_0^{\pi} [F_{1x}(x-\xi) - F_{1x}(x+\xi)]\,V_2(\xi,+0)\,d\xi \\ &= -\tfrac{1}{2}\int_0^{\pi} [F_{1\xi}(x+\xi) + F_{1\xi}(x-\xi)]\left(\frac{\pi-\xi}{\pi}\right)d\xi \\ &= F_1(x) - \frac{1}{2\pi}\int_0^{\pi} [F_1(x+\xi) + F_1(x-\xi)]\,d\xi \\ &= F_1(x) - \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} F_1(\zeta)\,d\zeta = F(x). \end{aligned} $$

In (9) we differentiate ${}_4V(x,t)$ with respect to t and, since $V_{2t}(x,+0) = 0$ for $0 < x < \pi$, we have ${}_4V_t(x,+0) = 0$.

We can check the partial differential equation as follows:

$$ \begin{aligned} {}_4V_{tt}(x,t) &= \tfrac{1}{2}\int_0^{\pi} [F_{1x}(x-\xi) - F_{1x}(x+\xi)]\,V_{2tt}(\xi,t)\,d\xi \\ &= \tfrac{1}{2}\int_0^{\pi} [F_{1x}(x-\xi) - F_{1x}(x+\xi)][C_1(t)\,V_{2\xi\xi} - C_2(t)\,V_2]\,d\xi \\ &= C_1(t)\tfrac{1}{2}\int_0^{\pi} [F_{1x}(x-\xi) - F_{1x}(x+\xi)]\,V_{2\xi\xi}\,d\xi - C_2(t)\,{}_4V(x,t) \\ &= C_1(t)\tfrac{1}{2}\int_0^{\pi} [F_{1xxx}(x-\xi) - F_{1xxx}(x+\xi)]\,V_2\,d\xi - C_2(t)\,{}_4V(x,t) \\ &= C_1(t)\,{}_4V_{xx}(x,t) - C_2(t)\,{}_4V(x,t). \end{aligned} $$

It is obvious that we can make a formal verification of ${}_5V(x,t)$ in the same manner.

We can summarize our results as follows:

THEOREM: *The transverse displacement function $V(x,t)$ can be expressed by means of the solutions ${}_iV(x,t)$, which are evaluated in (II—3.9–3.12), in terms of the auxiliary functions $V_1(x,t)$ and $V_2(x,t)$, defined in (II—3.1–3.2). The result can be written:*

$$ \begin{aligned} V(x,t) = &-\tfrac{1}{2}\int_0^t\int_{-\pi}^{\pi} P(x-\xi,\tau)\,W_{\xi\xi}(\xi,t,\tau)\,d\xi\,d\tau \\ &+ \int_0^t C_3(\tau)C_1(\tau)\,W_{xxx}(x,t,\tau)\,d\tau \\ &- \int_0^t C_4(\tau)C_1(\tau)\,W_{xxx}(\pi-0,t,\tau)\,d\tau \\ &+ \tfrac{1}{2}\int_{-\pi}^{\pi} F_1(x-\xi)\,V_{2\xi}(\xi,t)\,d\xi \\ &+ \tfrac{1}{2}\int_{-\pi}^{\pi} F_2(x-\xi)\,V_{1\xi}(\xi,t)\,d\xi, \end{aligned} $$

where

$$W(x, t, \tau) = -\int_{-\pi}^{\pi} [V_1(x-\xi, t)\, V_2(\xi, \tau) - V_1(x-\xi, \tau)\, V_2(\xi, t)]\, d\xi,$$

and both V_1 and V_2 are defined by extension in $(-2\pi, 2\pi)$ as odd and periodic with period 2π.

III. SURVEY OF THE FINITE FOURIER TRANSFORMATION

We have shown that the theory of ordinary differential equations can be applied to boundary value problems by means of the finite sine transformation and the finite cosine transformation. The introduction of the concept of a fundamental set of solutions of an ordinary differential equation enables us to make a resolution of the boundary problem into auxiliary problems.

In applying the finite sine or cosine transformations to a boundary value problem, the boundary conditions are involved automatically. Hence, if the endpoint values that are introduced by the transformation are not stated in the problem, we shall be confronted later with an integro-differential equation.

An examination of (I—3, 4) shows that this difficulty is avoided if we apply our transformations to problems which satisfy the following conditions: (a) The partial differential equation must be linear, and can have only coefficients which are constants or functions of time. This parallels the restriction of the use of the Laplace transformation, where in general only coefficients of the space variable can be present in the partial differential equation. (b) The boundary conditions must be symmetric. That is, for example, in the case of a boundary problem with a linear partial differential equation of the second order, it is necessary that either the function alone or else its first derivative alone is defined at both endpoints. However, if the endpoint conditions are asymmetrical, that is, for example, the function is defined as x approaches one end of the interval $(0, \pi)$ while its derivative is defined as x approaches the other end, then a sine or cosine transformation of odd order can be used. (c) The linear partial differential equation is restricted to even ordered derivatives with respect to the space variables, and the order of the derivatives in the boundary conditions must be either all even or all odd.

In spite of these restrictions there are many engineering problems[10] which can be solved by our method; an example of a type of problem not suited to our transformations is that of determining the transverse vibrations of a built-in cantilever beam.

[10] S. Timoshenko, *Vibration Problems in Engineering* (D. Van Nostrand Co., New York, 1937).

Calendar of Meetings

November

12–13 American Physical Society, Evanston, Illinois
15–16 American Institute of Chemical Engineers, Pittsburgh, Pennsylvania
26–27 Central Association of Science and Mathematics Teachers, Chicago, Illinois

December

27 American Physical Society, Pasadena, California

January

13–15 American Physical Society, New York, New York
13–15 American Association of Physics Teachers, New York, New York